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Differential Game Theory Application to Intelligent Missile Guidance

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ABSTRACT

In this report the application of the differential game theory to missile guidance problem is considered. Interceptor/target relative kinematics equations for a 3-D engagement are derived in state space form suitable for implementing the feedback guidance through minimisation-maximisation of the performance index. This performance index is a generalisation of that utilised by previous researchers in this field and includes, in addition to the miss-distance term, other terms involving interceptor/target relative velocity terms. This latter inclusion allows the designer to influence the engagement trajectories so as to aid both the intercept and evasion strategies. Closed form expressions are derived for the Riccati differential equations and the feedback guidance gains. A link between the differential game theory based guidance and the optimal guidance, the proportional navigation (PN) and the augmented PN (APN) guidance is established.
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Differential Game Theory Application to Intelligent Missile Guidance

Executive Summary

This report considers the application of the differential game theory to the missile guidance problem. The scenario considered involves engagement between an attacker (interceptor/pursuer) and a target (evader), where the objective of the former is to execute a strategy (manoeuvre) so as to achieve intercept with the target, while the objective of the latter is to execute a strategy (manoeuvre) so as to evade the attacker and avoid or at least delay the intercept. Differential game approach enables guidance strategies to be derived for both the attacker and the target so that objectives of both parties are satisfied.

Interceptor/target relative kinematics equations for a 3-D engagement are derived in state space form suitable for implementing feedback guidance laws through minimisation-maximisation of the performance index incorporating the game theory based objectives. This performance index is a generalisation of that utilised by previous researchers in this field and includes, in addition to the miss-distance term, other terms involving interceptor/target relative velocity terms. This latter inclusion allows the designer to influence the engagement trajectories so as to aid both the intercept and evasion strategies. Closed form expressions are derived for the Riccati differential equations and the feedback gains that allow the guidance of the interceptor and the target to be implemented. A link between the differential game theory based guidance and the optimal guidance, the proportional navigation (PN) and the augmented PN (APN) guidance is established.

The game theory based guidance technique proposed in this report provides a useful tool to study vulnerabilities of existing weapons systems against current and future threats that may incorporate 'intelligent' guidance or for enhancing the capability of future weapons by implementing (game theory based) intelligent guidance. Further research is required in this area in order to evaluate the performance of the game theoretic guidance in realistic combat environments.

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Nomenclature

$\mathbf{x_i}$: is the x-position of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{y_i}$: is the y-position of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{z_i}$: is the z-position of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{u_i}$: is the x-velocity of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{v_i}$: is the y-velocity of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{w_i}$: is the z-velocity of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{a_{x_i}}$: is the x-acceleration of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{a_{y_i}}$: is the y-acceleration of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{a_{z_i}}$: is the z-acceleration of vehicle $\mathbf{i}$ in fixed axis.
$\mathbf{x_{ij} = x_i - x_j}$: is the x-position of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{y_{ij} = y_i - y_j}$: is the y-position of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{z_{ij} = z_i - z_j}$: is the z-position of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{u_{ij} = u_i - u_j}$: is the x-velocity of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{v_{ij} = v_i - v_j}$: is the y-velocity of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{w_{ij} = w_i - w_j}$: is the z-velocity of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{a_{x_{ij}} = a_{x_i} - a_{x_j}}$: is the x-acceleration of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{a_{y_{ij}} = a_{y_i} - a_{y_j}}$: is the y-acceleration of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\mathbf{a_{z_{ij}} = a_{z_i} - a_{z_j}}$: is the z-acceleration of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\underline{\mathbf{x}}_\mathbf{i} = \begin{pmatrix} \mathbf{x_i} & \mathbf{y_i} & \mathbf{z_i} \end{pmatrix}^\mathbf{T}$: is the (3x1) position vector of vehicle $\mathbf{i}$ in fixed axis.
$\underline{\mathbf{u}}_\mathbf{i} = \begin{pmatrix} \mathbf{u_i} & \mathbf{v_i} & \mathbf{w_i} \end{pmatrix}^\mathbf{T}$: is the (3x1) velocity vector of vehicle $\mathbf{i}$ in fixed axis.
$\underline{\mathbf{a}}_\mathbf{i} = \begin{pmatrix} \mathbf{a_{x_i}} & \mathbf{a_{y_i}} & \mathbf{a_{z_i}} \end{pmatrix}^\mathbf{T}$: is the (3x1) acceleration vector of vehicle $\mathbf{i}$ in fixed axis.
$\underline{\mathbf{x}}_\mathbf{ij} = \begin{pmatrix} \mathbf{x_{ij}} & \mathbf{y_{ij}} & \mathbf{z_{ij}} \end{pmatrix}^\mathbf{T}$: is the (3x1) position vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\underline{\mathbf{u}}_\mathbf{ij} = \begin{pmatrix} \mathbf{u_{ij}} & \mathbf{v_{ij}} & \mathbf{w_{ij}} \end{pmatrix}^\mathbf{T}$: is the (3x1) velocity vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.
$\underline{\mathbf{a}}_\mathbf{ij} = \begin{pmatrix} \mathbf{a_{x_{ij}}} & \mathbf{a_{y_{ij}}} & \mathbf{a_{z_{ij}}} \end{pmatrix}^\mathbf{T} = \underline{\mathbf{a}}_\mathbf{i} - \underline{\mathbf{a}}_\mathbf{j}$: is the (3x1)acceleration vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ w.r.t in fixed axis.
$\underline{\mathbf{y}}_\mathbf{ij} = \begin{pmatrix} \underline{\mathbf{x}}_\mathbf{ij} & \underline{\mathbf{u}}_\mathbf{ij} \end{pmatrix}^\mathbf{T}$: is a (6x1) relative state vector between vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ defined w.r.t. the fixed axis.
$\mathbf{A}$: is a (6x6) state coefficient matrix.
$\mathbf{B}$: is a (6x3) control (input) coefficient matrix.
$\mathbf{S}$: is a 6x6 final state function penalty weighting matrix.
$\mathbf{I}$: is a 3x3 unity matrix.
$\mathbf{Q}$: is a 6x6 state function penalty weighting matrix.
$\mathbf{R_1}$: is a 3x3 pursuer's demanded acceleration function penalty weighting matrix.
$\mathbf{R_2}$: is a 3x3 evader's demanded acceleration function penalty weighting matrix.

$\mathbf{V}\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right)$:	is the optimum performance index/objective function.
$\mathbf{H}\left(\underline{\mathbf{y}}_{12}, \underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right)$:	is the Hamiltonian
$\mathbf{P}$:	is the Riccati matrix equation solution.
$\mathbf{T} = (\mathbf{t_f} - \mathbf{t})$:	is the time-to-go.
$\mathbf{K}_1^{\mathbf{p}}, \mathbf{K}_1^{\mathbf{d}}$:	are Interceptor (pursuer) state feedback guidance gains.
$\mathbf{K}_2^{\mathbf{e}}, \mathbf{K}_2^{\mathbf{d}}$:	are Target (evader) state feedback guidance gains.
$\underline{\xi}(\mathbf{t}) = \underline{\eta}(\mathbf{T})$:	is the Riccati vector equations solution.

1. Introduction

TACTICAL missiles have been in use since WWII and their guidance systems have progressively evolved from those employing proportional navigation (PN) and augmented proportional navigation (APN) to those employing optimal guidance (OG) and game theoretic guidance (GTG). One reason for this development is the fact that the implementation hardware for the guidance system has evolved and now offers greater flexibility to the system designer to implement advanced algorithms for missile navigation, guidance and control. Developments in the area of IR /RF missile-borne strap-down seekers, strap-down navigation instrumentation, and air-borne processors have prompted the guidance engineers to explore techniques that are more suited for the evolving and relatively more complex battlefield scenarios. With the advent of state estimation techniques such as the Kalman Filter and others, it is now feasible to implement the OG, GTG and GTG +AI 'intelligent' guidance on practical weapon systems.

It is noteworthy that the PN and APN are still being used in a number of modern missile guidance systems and give good performance. PN and APN performance has been studied by anumber of authors [1-5]. It is also interesting to note that, as shown in [5], both PN and APN guidance can be derived using the optimum control theory and state space representation of the interceptor and target kinematics. Thus we may regard both PN and APN as special cases of OG and GTG; this connection is further explored in this report. The desire to reduce weapon life-cycle cost, and at the same time extend the operational envelope to cope with complex engagement scenarios, that require the capability to adapt to adversary's 'intelligent' engagement tactics, it is necessary to consider OG and GTG guidance approaches for future tactical missiles. Augmentation of these guidance techniques with those that have evolved in the field of "intelligent systems" (e.g. AI, Fuzzy-Neural Nets, Bayesian and others), also need to be considered.

Application of the differential game theory to missile guidance has been considered by a number of authors [6-13]. Shinar, et.al., [7] presented an analysis of a complex combat scenario involving two parties (two aircraft), both equipped with interceptor missiles. The objective of each party was to shoot down the opponent's aircraft without their own aircraft being intercepted (hit) by their opponent's missile. Such a scenario, where the strategy of each party is for its missile to intercept the opponent's aircraft and perform evasion manoeuvres of its own aircraft so as to avoid being hit by opponents' missile, is a typical engagement scenario where the game theoretic approach to missile guidance can be used. Shinar refers to this situation as "non-cooperative differential game" which can also be classed as a "game of a kind". An encounter between the two aircraft (blue and red), under the above conditions results in one of the following outcomes:

a. Win for blue- (red alone is shot down)
b. Win for red- (blue alone is shot down)
c. Mutual kill- (both red and blue are shot down)
d. Draw- (both red and blue escape)

Shinar goes on to consider further combat strategies that can arise out of the above scenario and suggests the application of artificial intelligence (AI) augmentation to the differential game guidance. This latter aspect can be considered from the perspective of augmenting the GTG with a 'rule-based' AI - for switching performance index weighting parameters and/or for applying additional manoeuvres to evade the pursuer. In this report our main focus will be on the formulation and solution of the GTG problem

involving two parties where the objective of one party (pursuer) is to implement a strategy to intercept, while the objective of the other (evader) is to implement a strategy to evade the former. Ben-Asher, et.al., [8] considered the application of the differential game theory to missile guidance and utilised the optimum quadratic performance index approach in order to derive strategies (in terms of guidance acceleration commands) for the pursuer and the evader in a two party game scenario. The approach was based on defining the interceptor and target kinematics in linear state space form and the performance index as a quadratic functional of states and controls (the so called LQ-optimisation problem). The above authors considered engagement kinematics in 2-D and the performance index was tailored to that problem; in this report we have generalised the problem to 3-D. The GTG problem can be stated in a general form as follows. Given the following state space (relative kinematics) equations:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{y}}_{12} = \mathbf{A}\,\underline{\mathbf{y}}_{12} + \mathbf{B}\left(\underline{\mathbf{a}}_1^{\mathbf{p}} - \underline{\mathbf{a}}_2^{\mathbf{e}}\right) + \mathbf{B}\left(\underline{\mathbf{a}}_1^{\mathbf{d}} - \underline{\mathbf{a}}_2^{\mathbf{d}}\right); \qquad \underline{\mathbf{y}}_{12}(\mathbf{0}) = \underline{\mathbf{y}}_{12_0} \tag{1.1}$$

Where:

$\underline{\mathbf{y}}_{12} = \left(\underline{\mathbf{x}}_{12} \quad \underline{\mathbf{u}}_{12}\right)^{\mathbf{T}}$ **:** is the relative state (position and velocity) vector between the interceptor $\mathbf{1}$ and target $\mathbf{2}$ defined w.r.t. the fixed axis.

$\underline{\mathbf{x}}_{\mathbf{ij}} = \left(\mathbf{x}_{\mathbf{ij}} \quad \mathbf{y}_{\mathbf{ij}} \quad \mathbf{z}_{\mathbf{ij}}\right)^{\mathbf{T}}$ **:** is the (3x1) position vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.

$\underline{\mathbf{u}}_{\mathbf{ij}} = \left(\mathbf{u}_{\mathbf{ij}} \quad \mathbf{v}_{\mathbf{ij}} \quad \mathbf{w}_{\mathbf{ij}}\right)^{\mathbf{T}}$ **:** is the (3x1) velocity vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.

$\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right)$**:** are the commanded acceleration vectors respectively of the interceptor $\mathbf{1}$ (pursuer) and the target $\mathbf{2}$ (evader).

$\left(\underline{\mathbf{a}}_1^{\mathbf{d}}, \underline{\mathbf{a}}_2^{\mathbf{d}}\right)$**:** are the additional (pre-specified) acceleration vectors respectively of the interceptor and the target. These are included to admit additional evasion and/or pursuit manoeuvres that the parties might implement.

The guidance problem is to compute the interceptor and the target accelerations $\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right)$, such that the optimum performance index $\mathbf{V}^*\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right)$ is given by:

$$\mathbf{V}^*\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right) = \underset{\underline{\mathbf{a}}_1^{\mathbf{p}}}{\mathbf{Min}}\ \underset{\underline{\mathbf{a}}_2^{\mathbf{e}}}{\mathbf{Max}}\ \mathbf{V}\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right) \tag{1.2}$$

Where:

$$\mathbf{V}\left(\underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right) = \frac{\mathbf{1}}{\mathbf{2}}\underline{\mathbf{y}}_{12}^{\mathbf{T}}\mathbf{S}\,\underline{\mathbf{y}}_{12}\Big|_{\mathbf{t}=\mathbf{t_f}} + \frac{\mathbf{1}}{\mathbf{2}}\int_{\mathbf{0}}^{\mathbf{t_f}}\left\{\underline{\mathbf{y}}_{12}^{\mathbf{T}}\mathbf{Q}\,\underline{\mathbf{y}}_{12} + \underline{\mathbf{a}}_1^{\mathbf{p}^{\mathbf{T}}}\mathbf{R}_1\,\underline{\mathbf{a}}_1^{\mathbf{p}} - \underline{\mathbf{a}}_2^{\mathbf{e}^{\mathbf{T}}}\mathbf{R}_2\,\underline{\mathbf{a}}_2^{\mathbf{e}}\right\}\mathbf{dt} \tag{1.3}$$

$\mathbf{S}, \mathbf{Q}$ **:** are positive semi-definite matrices that define the penalty weightings on the states.

$\mathbf{R}_1, \mathbf{R}_2$ **:** are positive definite matrices that define the penalty weightings on the inputs.

The authors [8] solved the problem (1.1) – (1.3) for a special case of 2-D with a performance index consisting of miss distance squared term only. In this current report we shall consider engagement in 3-D (azimuth and elevation plane engagement) and define a performance index that incorporates the miss distance term as well as additional terms consisting of relative (velocity) states that may allow us to shape the engagement trajectory and more effectively deal with large heading errors, unfavourable engagement geometries

and severe interceptor and target manoeuvres. It must be pointed out that, in general, the game outcome depends on which of the parties 'plays first' [7, 8]. We therefore assume that the game evolves such that both parties apply 'strategies' (guidance commands) that satisfies the 'saddle point' solution for the performance index; that is both parties must utilise optimum strategies, in which case:

$$\mathbf{V}^*\left(\underline{\mathbf{a}}_1^p, \underline{\mathbf{a}}_2^e\right) = \underset{\underline{\mathbf{a}}_1^p}{\mathbf{Min}}\ \underset{\underline{\mathbf{a}}_2^e}{\mathbf{Max}}\ \mathbf{V}\left(\underline{\mathbf{a}}_1^p, \underline{\mathbf{a}}_2^e\right) = \underset{\underline{\mathbf{a}}_2^e}{\mathbf{Max}}\ \underset{\underline{\mathbf{a}}_1^p}{\mathbf{Min}}\ \mathbf{V}\left(\underline{\mathbf{a}}_1^p, \underline{\mathbf{a}}_2^e\right) \quad (1.4)$$

Both OG and GTG are derived, based on defining an objective function (performance index PI) that is a function of system states and controls that have to be minimised (w.r.t to the pursuer's guidance commands) and maximised (w.r.t the evader's guidance commands). In the case of the OG, which can be regarded as a special case (a subset) of the GTG, it is assumed that the target does not implement the optimum evasion strategy while the interceptor implements the optimum intercept strategy and does not perform any additional manoeuvres. Thus, with a slight modification, the objective function (1.3) may be used to derive the OG law. The optimisation problem for linear systems that use quadratic objective function (the LQ-control problem) was considered by Sage, A.P [14]; the solution to this problem was obtained through minimisation of a Hamiltonian type function. This yields a linear state feedback control law involving 'gain' terms derived from the, well known, Riccati differential equations (see section 3). This approach has been adopted in this report to solve the GTG and the OG problems. In both cases, closed form solution of the Riccati equation and the resulting feedback gains are derived as a function of time-to-go. While, the emphasis, in this report is on two party games involving one interceptor against one target, the development of the kinematics equations and the guidance law derivation is general enough to be extended to multi-party game situation [7, 10].

Intercept and evasion strategies, implemented by the parties involved, are based on their knowledge of relative states (i.e. the parties learn from the environment) and on optimisation of the objective function (i.e. the decision-making criteria). Rule based AI methodology can also be implemented that will allow adaptively changing the PI weightings and/or implementing additional manoeuvres $\left(\underline{\mathbf{a}}_1^d, \underline{\mathbf{a}}_2^d\right)$. In view of these attributes, we shall refer to the GTG as 'intelligent' guidance.

Section 2 of this report presents the development of a 3-D engagement kinematics model in state space form. Section 3 of the report presents the formulation of the objective function and the solution of the GTG problem. Solutions of the matrix Riccati differential equations MRDE and the vector Riccati differential equations VRDE are considered in section 4, along the feedback implementation of the guidance law. Relationships between the OG and GTG and the conventional PN and APN guidance are explored in sections 5 and 6. Section 7 contains conclusions resulting from the material presented in this report.

2. Development of the Engagement Kinematics Model

Typical two vehicle engagement geometry is shown in Figure 2.1 ($\mathbf{T}$ is the target and $\mathbf{I}$ is the interceptor); this scenario may be extended to the case of $\mathbf{m}$ targets and $\mathbf{n}$ interceptors; that is $\mathbf{N} = \mathbf{n} + \mathbf{m}$ vehicles. The engagement kinematics model in this report is

derived with this in mind. The motion (kinematics) of each vehicle can be described by a set of first order differential equations representing states of the vehicles (i.e. position, velocity and acceleration) defined in fixed (e.g. inertial) frame. The kinematics equations may be written as:

$$\frac{d}{dt}x_i = u_i;\ \frac{d}{dt}y_i = v_i;\ \frac{d}{dt}z_i = w_i \tag{2.1}$$

$$\frac{d}{dt}u_i = a_{x_i};\ \frac{d}{dt}v_i = a_{y_i};\ \frac{d}{dt}w_i = a_{z_i} \tag{2.2}$$

Where:
The above variables are functions of time t.
x_i : is the x-position of vehicle i in fixed axis.
y_i : is the y-position of vehicle i in fixed axis.
z_i : is the z-position of vehicle i in fixed axis.
u_i : is the x-velocity of vehicle i in fixed axis.
v_i : is the y-velocity of vehicle i in fixed axis.
w_i : is the z-velocity of vehicle i in fixed axis.
a_{x_i} : is the x-acceleration of vehicle i in fixed axis.
a_{y_i} : is the y-acceleration of vehicle i in fixed axis.
a_{z_i} : is the z-acceleration of vehicle i in fixed axis.

Note that a 'flat earth' assumption is made and that Z-axis is assumed positive 'down'.

2.1 Relative Engage Kinematics of n vs m Vehicles.

We now consider relative state variables of the vehicles; the kinematics equations may be written as follows:

$$\frac{d}{dt}x_{ij} = u_{ij};\ \frac{d}{dt}y_{ij} = v_{ij};\ \frac{d}{dt}z_{ij} = w_{ij} \tag{2.3}$$

$$\frac{d}{dt}u_{ij} = a_{x_{ij}} = a_{x_i} - a_{x_j};\ \frac{d}{dt}v_{ij} = a_{y_{ij}} = a_{y_i} - a_{y_j};\ \frac{d}{dt}w_i = a_{z_{ij}} = a_{z_i} - a_{z_j} \tag{2.4}$$

Where:
$x_{ij} = x_i - x_j$: is the x-position of vehicle i w.r.t j in fixed axis.
$y_{ij} = y_i - y_j$: is the y-position of vehicle i w.r.t j in fixed axis.
$z_{ij} = z_i - z_j$: is the z-position of vehicle i w.r.t j in fixed axis.
$u_{ij} = u_i - u_j$: is the x-velocity of vehicle i w.r.t j in fixed axis.
$v_{ij} = v_i - v_j$: is the y-velocity of vehicle i w.r.t j in fixed axis.
$w_{ij} = w_i - w_j$: is the z-velocity of vehicle i w.r.t j in fixed axis.
$a_{x_{ij}} = a_{x_i} - a_{x_j}$: is the x-acceleration of vehicle i w.r.t j in fixed axis.
$a_{y_{ij}} = a_{y_i} - a_{y_j}$: is the y-acceleration of vehicle i w.r.t j in fixed axis.
$a_{z_{ij}} = a_{z_i} - a_{z_j}$: is the z-acceleration of vehicle i w.r.t j in fixed axis.

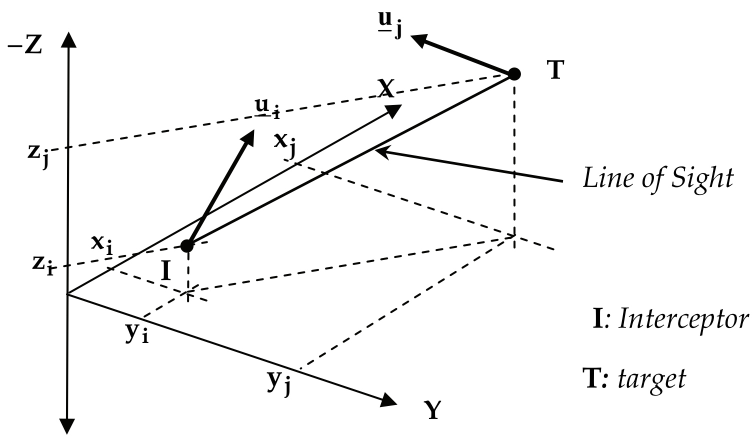

Figure 2.1- Interceptor/Target Engagement Geometry

In the context of the engagement between an interceptor against a target (2-party engagement), we may regard suffix $\mathbf{i}$ to represent the interceptor and suffix $\mathbf{j}$ to represent the target. In the derivation of the optimal guidance law it will be useful to represent the above equations in vector/matrix notation.

2.1.1 Vector/Matrix Representation

We can write equations (2.1), (2.2) as:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{x}}_{\mathbf{i}} = \underline{\mathbf{u}}_{\mathbf{i}}; \quad \frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{u}}_{\mathbf{i}} = \underline{\mathbf{a}}_{\mathbf{i}} \qquad (2.5)$$

Similarly we can write the relative kinematics equations (2.3), (2.4) as:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{x}}_{\mathbf{ij}} = \underline{\mathbf{u}}_{\mathbf{ij}}; \quad \frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{u}}_{\mathbf{ij}} = \underline{\mathbf{a}}_{\mathbf{i}} - \underline{\mathbf{a}}_{\mathbf{j}} \qquad (2.6)$$

Where:

$\underline{\mathbf{x}}_{\mathbf{i}} = (\mathbf{x}_{\mathbf{i}} \quad \mathbf{y}_{\mathbf{i}} \quad \mathbf{z}_{\mathbf{i}})^{\mathbf{T}}$ **:** is the (3x1) position vector of vehicle $\mathbf{i}$ in fixed axis.

$\underline{\mathbf{u}}_{\mathbf{i}} = (\mathbf{u}_{\mathbf{i}} \quad \mathbf{v}_{\mathbf{i}} \quad \mathbf{w}_{\mathbf{i}})^{\mathbf{T}}$ **:** is the (3x1) velocity vector of vehicle $\mathbf{i}$ in fixed axis.

$\underline{\mathbf{a}}_{\mathbf{i}} = \left(\mathbf{a}_{\mathbf{x_i}} \quad \mathbf{a}_{\mathbf{y_i}} \quad \mathbf{a}_{\mathbf{z_i}}\right)^{\mathbf{T}}$ **:** is the (3x1) acceleration vector of vehicle $\mathbf{i}$ in fixed axis.

$\underline{\mathbf{x}}_{\mathbf{ij}} = \left(\mathbf{x}_{\mathbf{ij}} \quad \mathbf{y}_{\mathbf{ij}} \quad \mathbf{z}_{\mathbf{ij}}\right)^{\mathbf{T}}$ **:** is the (3x1) position vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.

$\underline{\mathbf{u}}_{\mathbf{ij}} = \left(\mathbf{u}_{\mathbf{ij}} \quad \mathbf{v}_{\mathbf{ij}} \quad \mathbf{w}_{\mathbf{ij}}\right)^{\mathbf{T}}$ **:** is the (3x1) velocity vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.

$\underline{\mathbf{a}}_{\mathbf{ij}} = \left(\mathbf{a}_{\mathbf{x_{ij}}} \quad \mathbf{a}_{\mathbf{y_{ij}}} \quad \mathbf{a}_{\mathbf{z_{ij}}}\right)^{\mathbf{T}} = \underline{\mathbf{a}}_{\mathbf{i}} - \underline{\mathbf{a}}_{\mathbf{j}}$ **:** is the (3x1) acceleration vector of vehicle $\mathbf{i}$ w.r.t $\mathbf{j}$ in fixed axis.

Equation (2.6) may be combined together to give us:

$$\frac{\mathbf{d}}{\mathbf{dt}}\begin{bmatrix}\underline{\mathbf{x}}_{\mathbf{ij}}\\ \underline{\mathbf{u}}_{\mathbf{ij}}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\underline{\mathbf{x}}_{\mathbf{ij}}\\ \underline{\mathbf{u}}_{\mathbf{ij}}\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}\underline{\mathbf{a}}_{\mathbf{i}}-\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}\underline{\mathbf{a}}_{\mathbf{j}} \tag{2.7}$$

This can be written as:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{y}}_{\mathbf{ij}}=\mathbf{A}\,\underline{\mathbf{y}}_{\mathbf{ij}}+\mathbf{B}\,\underline{\mathbf{a}}_{\mathbf{i}}-\mathbf{B}\,\underline{\mathbf{a}}_{\mathbf{j}} \tag{2.8}$$

Where:

$\underline{\mathbf{y}}_{\mathbf{ij}}=\left(\underline{\mathbf{x}}_{\mathbf{ij}} \quad \underline{\mathbf{u}}_{\mathbf{ij}}\right)^{\mathbf{T}}$ **:** is a (6x1) relative state vector between vehicle $\mathbf{i}$ and vehicle $\mathbf{j}$ defined w.r.t. the fixed axis.

$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$ **:** is a (6x6) state coefficient matrix.

$\mathbf{B}=\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}$ **:** is a (6x3) control (input) coefficient matrix., $\mathbf{I}$ is the identity matrix.

Remarks

In this report the guidance algorithm is derived based on linear engagement kinematics defined in the fixed (e.g. inertial axis). There are thus implied assumptions in the derivation: (a) that the body attitude does not vary significantly during the engagement, and (b) that the guidance commands are implemented in the fixed axis. However, in most engagements the body attitude may vary significantly during the engagement and more importantly the guidance commands are applied in the body axis and require the guidance commands (generated in fixed axis) to be transformed to body axis. For testing and performance of the guidance strategies derived in this report, a non-linear engagement kinematics model should be used for the simulation model. Such a simulation model was derived by the author and is given in [11], which will be referred to as the Integrated Navigation, Guidance and Control Test Bed (INGC-Test Bed). The guidance commands should be applied in the vehicle body axis (through appropriate transformation) which accounts for changes in body attitude; also the autopilot dynamics and effects such as the gravity and vehicle lateral acceleration-limits should be included in the simulation model. Several authors [5] have included autopilot lags in the guidance law derivations and the material presented in this report may be extended to that case.

3. Optimum Interceptor/Target Guidance for a 2-Party Game Scenario

We shall first of all address the problem of a target that is being engaged by an interceptor and wishes to implement a guidance strategy to avoid intercept; while the interceptor endeavours to implement a strategy to effect an intercept. The respective strategies are implemented via the application of guidance acceleration commands. A scenario of this type will be referred to as a 2-party game.

The interceptor $\mathbf{i=1}$ utilises its guidance input command $\underline{\mathbf{a}}_1$ to effect the intercept of the target, this is specified by $\underline{\mathbf{a}}_1^{\mathbf{p}}$; in addition it may, if required, perform an additional pre-specified manoeuvre $\underline{\mathbf{a}}_1^{\mathbf{d}}$. The total interceptor acceleration in this case may be written as:

$$\underline{\mathbf{a}}_1 = \underline{\mathbf{a}}_1^{\mathbf{p}} + \underline{\mathbf{a}}_1^{\mathbf{d}} \tag{3.1}$$

The target $(\mathbf{j=2})$ on the other hand, applies an evasive manoeuvre $\underline{\mathbf{a}}_2^{\mathbf{e}}$, and an additional pre-specified (disturbance) manoeuvre $\underline{\mathbf{a}}_2^{\mathbf{d}}$; this latter component could be, for example, a random manoeuvre or a manoeuvre of a periodical wave form. With these manoeuvres, the total evader acceleration is of the form (see equations (2-8)):

$$\underline{\mathbf{a}}_2 = \underline{\mathbf{a}}_2^{\mathbf{e}} + \underline{\mathbf{a}}_2^{\mathbf{d}} \tag{3.2}$$

The modified form of the kinematics equation (2.8), including the above manoeuvres may be written as:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{y}}_{12} = \mathbf{A}\,\underline{\mathbf{y}}_{12} + \mathbf{B}\left(\underline{\mathbf{a}}_1^{\mathbf{p}} + \underline{\mathbf{a}}_1^{\mathbf{d}}\right) - \mathbf{B}\left(\underline{\mathbf{a}}_2^{\mathbf{e}} + \underline{\mathbf{a}}_2^{\mathbf{d}}\right) \tag{3.3}$$

The object of the 2-party game is to compute the evasion and pursuit guidance (strategies) commands $\underline{\mathbf{a}}_2^{\mathbf{e}}$, $\underline{\mathbf{a}}_1^{\mathbf{p}}$ that satisfy specified criteria. For this report we shall consider the application of the differential game and the optimum control principles to derive both the evasion and pursuit guidance commands (strategies).

3.1 Construction of the Differential Game Objective Function

In the context of missile guidance, differential game ensues if the interceptor's objective is to utilise a strategy (i.e. execute guidance commands) that results in an intercept with the target, while the objective of the target is to utilise a strategy (i.e. executes guidance command) so as to delay or totally avoid intercept and eventually escape from the interceptor. Here, intercept implies that the minimum separation between the target and the interceptor is small or preferably zero. In formulating the DG guidance problem, the following assumptions are made:

(a) Both parties have all the necessary information of the states/relative states, with respect to each other, to enable the parties to implement the necessary guidance laws. Countermeasures designed to conceal states of parties involved are not considered.

(b) If it is assumed that a seeker/target tracker is used to construct system states from seeker information (e.g. utilising a Kalman Filter). However, for the purpose of our current considerations it is assumed the system states (or state estimates) are exact. Later studies will include a state estimator.

(c) The maximum and the minimum accelerations (both lateral and longitudinal) commanded and consequently achievable by the vehicles involved in the game are limited. For our current derivation constraints on the accelerations are considered to be "soft"; that is, the change in acceleration and acceleration rate is gradual in the neighbourhood of maximum/minimum values. This type of constraint can be

implemented in the objective function through the use of 'penalty weightings' associated with the demanded acceleration, and leads to a relatively easier solution and implementation of the guidance law.

(d) In the derivation of the guidance laws, autopilot lag is ignored. However, this may be included in the actual simulation studies in order to assess the guidance performance including the autopilot.

Under the above assumptions we can proceed to construct the objective function (performance index) that the parties to the game will need to minimise or maximise in order to derive their respective strategies. Note that the interceptor's objective is to minimise the relative separation (miss distance), while that of the target is to maximise the miss distance. The objective function, therefore, must include interceptor/target states that represent the miss distance as well as other states that influence this. In the past, most authors (8, 10) have used miss distance (which is a function of relative position) and demanded accelerations terms to construct the objective function. In this report we shall, generalise the objective function by including terms in relative position and relative velocity (states) as well as demanded accelerations (input). Thus the objective function may be written as:

$$\begin{aligned} V\left(\underline{x}_{12},\underline{u}_{12},\underline{a}_1^p,\underline{a}_2^e\right) &= \left\{s_1\left(\underline{x}_{12}{}^T\underline{x}_{12}\right)+s_2\left(\underline{x}_{12}{}^T\underline{u}_{12}\right)+s_3\left(\underline{u}_{12}{}^T\underline{u}_{12}\right)\right\}_{t=t_f} \\ &+\int_0^{t_f}\left\{q_1\left(\underline{x}_{12}{}^T\underline{x}_{12}\right)+q_2\left(\underline{x}_{12}{}^T\underline{u}_{12}\right)+q_3\left(\underline{u}_{12}{}^T\underline{u}_{12}\right)+r_1\left(\underline{a}_1^{pT}\underline{a}_1^p\right)-r_2\left(\underline{a}_2^{eT}\underline{a}_2^e\right)\right\}dt \end{aligned} \tag{3.4}$$

Where:
$V(..)$: is the objective functional.
$\left(\underline{x}_{12}{}^T\underline{x}_{12}\right)$: is the square of the relative range (separation) between the interceptor and the target. $\left(\underline{x}_{12}{}^T\underline{u}_{12}\right)$: is the projection of the relative velocity on to the relative range.
$\left(\underline{u}_{12}{}^T\underline{u}_{12}\right)$: is the square of the relative velocity.
$\left(\underline{a}_1^{pT}\underline{a}_1^p\right)$: is the square of the interceptor (pursuer's) acceleration.
$\left(\underline{a}_2^{eT}\underline{a}_2^e\right)$: is the square of the target (evader's) acceleration.
(s_1,s_2,s_3): are penalty weighting on the final values of state functions.
(q_1,q_2,q_3): are penalty weighting on state functions.
(r_1,r_2): are penalty weighting on demanded accelerations.

By varying the relative values of the penalty weightings (s_i,q_i,r_i), constraints on system states and control can be implemented. Note that the objective function contains:

(a) Interceptor/target relative position term: $\left(\underline{x}_{12}{}^T\underline{x}_{12}\right)$; at the final time $t=t_f$ it is the miss distance squared.

(b) Interceptor/target relative position and velocity terms: $\left(\underline{x}_{12}{}^T\underline{u}_{12}\right)$ and $\left(\underline{u}_{12}{}^T\underline{u}_{12}\right)$ these terms represent engagement trajectory shaping terms.

(c) Interceptor/target demanded acceleration terms: $\left(\underline{\mathbf{a}}_1^{\mathbf{p}\,\mathbf{T}}\,\underline{\mathbf{a}}_1^{\mathbf{p}}\right)$ and $\left(\underline{\mathbf{a}}_2^{\mathbf{e}\,\mathbf{T}}\,\underline{\mathbf{a}}_2^{\mathbf{e}}\right)$; these terms allow soft constraints on controls (demanded accelerations) to be implemented.

(d) Relative values of penalty weightings: $(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3)$, $(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ and $(\mathbf{r}_1,\mathbf{r}_2)$ determine soft constraint boundaries on states and control variables.

The objective function $\mathbf{V}(..)$ as given in equation (3.4) can be minimised w.r.t $\underline{\mathbf{a}}_1^{\mathbf{p}}$ in order to derive the intercept guidance commands, and maximised w.r.t $\underline{\mathbf{a}}_2^{\mathbf{e}}$, by virtue of the negative sign associated with it, in order to derive the evasion guidance commands for the target. It will be convenient to write the objective function as:

$$\mathbf{V}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=\frac{1}{2}\left\|\underline{\mathbf{y}}_{12}\right\|_{\mathbf{S}}^2\Big|_{t=t_f}+\frac{1}{2}\int_0^{t_f}\left\{\left\|\underline{\mathbf{y}}_{12}\right\|_{\mathbf{Q}}^2+\left\|\underline{\mathbf{a}}_1^{\mathbf{p}}\right\|_{\mathbf{R}_1}^2-\left\|\underline{\mathbf{a}}_2^{\mathbf{e}}\right\|_{\mathbf{R}_2}^2\right\}\mathbf{dt} \tag{3.5}$$

Where:

$\mathbf{S}=\begin{bmatrix}2\mathbf{s}_1\mathbf{I} & \mathbf{s}_2\mathbf{I}\\ \mathbf{s}_2\mathbf{I} & 2\mathbf{s}_3\mathbf{I}\end{bmatrix}$**:** is a 6x6 final state function penalty weighting matrix. $\mathbf{I}$**:** is a 3x3 unity matrix.

$\mathbf{Q}=\begin{bmatrix}2\mathbf{q}_1\mathbf{I} & \mathbf{q}_2\mathbf{I}\\ \mathbf{q}_2\mathbf{I} & 2\mathbf{q}_3\mathbf{I}\end{bmatrix}$**:** is a 6x6 state function penalty weighting matrix.

$\mathbf{R}_1=\mathbf{r}_1\mathbf{I}$**:** is a 3x3 pursuer's demanded acceleration function penalty weighting matrix.

$\mathbf{R}_2=\mathbf{r}_2\mathbf{I}$**:** is a 3x3 evader's demanded acceleration function penalty weighting matrix.

$\|\underline{\alpha}\|_{\Lambda}^2\equiv\underline{\alpha}^{\mathbf{T}}\Lambda\,\underline{\alpha}$.

In this report we will consider the objective function with $\mathbf{Q}=\mathbf{0}$. The game theoretic guidance problem can be stated as that of minimising the objective function $\mathbf{V}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)$ w.r.t $\underline{\mathbf{a}}_1^{\mathbf{p}}$ and maximising it w.r.t $\underline{\mathbf{a}}_2^{\mathbf{e}}$, that is:

$$\underset{\underline{\mathbf{a}}_1^{\mathbf{p}}}{\mathbf{Min}}\;\underset{\underline{\mathbf{a}}_2^{\mathbf{e}}}{\mathbf{Max}}\;\mathbf{V}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=\underset{\underline{\mathbf{a}}_1^{\mathbf{p}}}{\mathbf{Min}}\;\underset{\underline{\mathbf{a}}_2^{\mathbf{e}}}{\mathbf{Max}}\left\{\frac{1}{2}\left\|\underline{\mathbf{y}}_{12}\right\|_{\mathbf{S}}^2\Big|_{t=t_f}+\frac{1}{2}\int_0^{t_f}\left(\left\|\underline{\mathbf{a}}_1^{\mathbf{p}}\right\|_{\mathbf{R}_1}^2-\left\|\underline{\mathbf{a}}_2^{\mathbf{e}}\right\|_{\mathbf{R}_2}^2\right)\mathbf{dt}\right\} \tag{3.6}$$

Remarks

For a minimum and a maximum of the objective function to exist, it is a requirement that the matrix $\mathbf{S}$ be at least positive semi-definite, and matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ be positive definite. That is, the determinants: $|\mathbf{S}|\geq\mathbf{0}$ and $|\mathbf{R}_1|>\mathbf{0}$, $|\mathbf{R}_2|>\mathbf{0}$. These conditions imply that (see Appendix A.1):

$\mathbf{s}_1,\mathbf{s}_2\geq\mathbf{0}$, and $\left(\mathbf{4s}_1\mathbf{s}_3-\mathbf{s}_2{}^2\right)\geq\mathbf{0}$, $\mathbf{s}_2$ can be positive or negative (3.7)

3.2 Solution of the Differential Game Guidance Problem

In this section we present the solution to the problem of optimisation of the objective function (3.6) subject to the condition that the kinematics equation (3.3) holds. In doing so we shall follow the technique described in [14]. This involves the construction of the

Hamiltonian, which is minimised w.r.t $\underline{\mathbf{a}}_1^{\mathbf{p}}$ and maximised w.r.t $\underline{\mathbf{a}}_2^{\mathbf{e}}$. The Hamiltonian may be written as:

$$\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=\frac{1}{2}\left\{\left(\underline{\mathbf{a}}_1^{\mathbf{p}\,\mathbf{T}}\mathbf{R}_1\underline{\mathbf{a}}_1^{\mathbf{p}}\right)-\left(\underline{\mathbf{a}}_2^{\mathbf{e}\,\mathbf{T}}\mathbf{R}_2\underline{\mathbf{a}}_2^{\mathbf{e}}\right)\right\}\cdots \\ +\underline{\lambda}\left(\mathbf{A}\,\underline{\mathbf{y}}_{12}+\mathbf{B}\left(\underline{\mathbf{a}}_1^{\mathbf{p}}+\underline{\mathbf{a}}_1^{\mathbf{d}}\right)-\mathbf{B}\left(\underline{\mathbf{a}}_2^{\mathbf{e}}+\underline{\mathbf{a}}_2^{\mathbf{d}}\right)\right) \tag{3.8}$$

Sufficient conditions for $\underset{\underline{\mathbf{a}}_1^{\mathbf{p}}}{\mathbf{Min}}\ \underset{\underline{\mathbf{a}}_2^{\mathbf{e}}}{\mathbf{Max}}\ \mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)$ are given by:

(a) $\frac{\partial}{\partial\underline{\mathbf{a}}_1^{\mathbf{p}}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=\mathbf{0}$ and $\frac{\partial}{\partial\underline{\mathbf{a}}_2^{\mathbf{e}}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=\mathbf{0}$

(b) $\frac{\partial}{\partial\underline{\mathbf{y}}_{12}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=-\dot{\underline{\lambda}}$

(c) The terminal condition for $\underline{\lambda}$ is given by: $\underline{\lambda}(\mathbf{t_f})=\mathbf{S}\,\underline{\mathbf{y}}_{12}(\mathbf{t_f})$.

The corresponding necessary conditions are

(a) $\frac{\partial^2}{\partial\underline{\mathbf{a}}_1^{\mathbf{p}\,2}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)\geq\mathbf{0}$ and $\frac{\partial^2}{\partial\underline{\mathbf{a}}_2^{\mathbf{e}\,2}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)\leq\mathbf{0}$

Now, applying the sufficiency conditions (a) to the Hamiltonian (3.8), we get:

$$\frac{\partial}{\partial\underline{\mathbf{a}}_1^{\mathbf{p}}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=\mathbf{R}_1\underline{\mathbf{a}}_1^{\mathbf{p}}+\mathbf{B}^{\mathbf{T}}\underline{\lambda}=\mathbf{0} \tag{3.9}$$

$$\frac{\partial}{\partial\underline{\mathbf{a}}_2^{\mathbf{e}}}\mathbf{H}\left(\underline{\mathbf{y}}_{12},\underline{\mathbf{a}}_1^{\mathbf{p}},\underline{\mathbf{a}}_2^{\mathbf{e}}\right)=-\mathbf{R}_2\underline{\mathbf{a}}_2^{\mathbf{e}}-\mathbf{B}^{\mathbf{T}}\underline{\lambda}=\mathbf{0} \tag{3.10}$$

Equations (3.9), (3.10) give us:

$$\underline{\mathbf{a}}_1^{\mathbf{p}}=-\mathbf{R}_1^{-1}\mathbf{B}^{\mathbf{T}}\underline{\lambda};\ \text{and}\ \ \underline{\mathbf{a}}_2^{\mathbf{e}}=-\mathbf{R}_2^{-1}\mathbf{B}^{\mathbf{T}}\underline{\lambda} \tag{3.11}$$

It is left to the reader to verify the necessary conditions. We are interested in constructing guidance commands that are functions of system relative kinematics states; we shall therefore assume that $\underline{\lambda}$ is of the form:

$$\underline{\lambda}=\mathbf{P}\,\underline{\mathbf{y}}_{12}+\underline{\xi} \tag{3.12}$$

Where:
$\mathbf{P}$: is a 6x6 matrix, which will be later shown to be the solution of the matrix Riccati differential equation (MRDE), and
$\underline{\xi}$: is a 6x1 vector, which will be later shown to be the solution of the vector Riccati differential equation (VRDE).

Thus:

$$\underline{\mathbf{a}}_1^{\mathbf{p}}=-\mathbf{R}_1^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\,\underline{\mathbf{y}}_{12}-\mathbf{R}_1^{-1}\mathbf{B}^{\mathbf{T}}\underline{\xi};\ \text{and}\ \ \underline{\mathbf{a}}_2^{\mathbf{e}}=-\mathbf{R}_2^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\,\underline{\mathbf{y}}_{12}-\mathbf{R}_2^{-1}\mathbf{B}^{\mathbf{T}}\underline{\xi} \tag{3.13}$$

We shall write these equations as:

$$\underline{\mathbf{a}}_1^{\mathbf{p}} = -\mathbf{K}_1^{\mathbf{p}}\,\underline{\mathbf{y}}_{12} - \mathbf{K}_1^{\mathbf{d}}\underline{\xi}\text{; and } \underline{\mathbf{a}}_2^{\mathbf{e}} = -\mathbf{K}_2^{\mathbf{e}}\,\underline{\mathbf{y}}_{12} - \mathbf{K}_2^{\mathbf{d}}\underline{\xi} \qquad (3.14)$$

Where:

$\mathbf{K}_1^{\mathbf{p}} = \mathbf{R}_1^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}$: is the state feedback gain for the pursuer; and $\mathbf{K}_1^{\mathbf{d}} = \mathbf{R}_1^{-1}\mathbf{B}^{\mathbf{T}}$: is the disturbance input gain of the pursuer.

$\mathbf{K}_2^{\mathbf{e}} = \mathbf{R}_2^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}$: is the state feedback gain for the evader; and $\mathbf{K}_2^{\mathbf{d}} = \mathbf{R}_2^{-1}\mathbf{B}^{\mathbf{T}}$: is the disturbance input gain of the evader.

Now applying the sufficiency condition (b), we get, using equation (3.8):

$$\frac{\partial}{\partial \underline{\mathbf{y}}_{12}}\mathbf{H}\left(\underline{\mathbf{y}}_{12}, \underline{\mathbf{a}}_1^{\mathbf{p}}, \underline{\mathbf{a}}_2^{\mathbf{e}}\right) = \mathbf{A}^{\mathbf{T}}\,\underline{\lambda} = -\dot{\underline{\lambda}} \qquad (3.15)$$

Where:

$\dot{\underline{\lambda}} = \dot{\mathbf{P}}\,\underline{\mathbf{y}}_{12} + \mathbf{P}\,\dot{\underline{\mathbf{y}}}_{12} + \dot{\underline{\xi}}$

Substituting for $\underline{\lambda}$, $\dot{\underline{\lambda}}$, and $\dot{\underline{\mathbf{y}}}_{12}$, and for $\underline{\mathbf{a}}_1^{\mathbf{p}}$ and $\underline{\mathbf{a}}_2^{\mathbf{e}}$, and after algebraic simplification (detailed derivation is given in Appendix A.2) it can be shown that equation (3.15) leads to:

$$\begin{aligned}&\left\{\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^{\mathbf{T}}\mathbf{P} - \mathbf{PB}\left(\mathbf{R}_1^{-1} - \mathbf{R}_2^{-1}\right)\mathbf{B}^{\mathbf{T}}\mathbf{P}\right\}\underline{\mathbf{y}}_{12}\cdots\\ &\qquad = \left\{-\dot{\underline{\xi}} - \left[\mathbf{A} - \mathbf{B}\left(\mathbf{R}_1^{-1} - \mathbf{R}_2^{-1}\right)\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]^{\mathbf{T}} - \mathbf{PB}\left(\underline{\mathbf{a}}_1^{\mathbf{d}} - \underline{\mathbf{a}}_2^{\mathbf{d}}\right)\right\}\end{aligned} \qquad (3.16)$$

Since the solution of equation (3.15) must hold for all $\underline{\mathbf{y}}_{12}$, a solution of this equation can be written as:

$$\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^{\mathbf{T}}\mathbf{P} - \mathbf{PB}\left(\mathbf{R}_1^{-1} - \mathbf{R}_2^{-1}\right)\mathbf{B}^{\mathbf{T}}\mathbf{P} = \mathbf{0} \qquad (3.17)$$

And:

$$-\dot{\underline{\xi}} - \left[\mathbf{A} - \mathbf{B}\left(\mathbf{R}_1^{-1} - \mathbf{R}_2^{-1}\right)\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]^{\mathbf{T}} - \mathbf{PB}\left(\underline{\mathbf{a}}_1^{\mathbf{d}} - \underline{\mathbf{a}}_2^{\mathbf{d}}\right) = \mathbf{0} \qquad (3.18)$$

With terminal conditions $\mathbf{P}(\mathbf{t_f}) = \mathbf{S}$, and $\underline{\xi}(\mathbf{t_f}) = \underline{\mathbf{0}}$. In the sequel, equation (3.17) will be referred to as the MRDE and equation (3.18) will be referred to as the VRDE.

4. Solution of the Riccati Differential Equations

4.1 Solution of the Matrix Riccati Differential Equations (MRDE)

In this section we shall consider deriving the solution of the MRDE (3.18). We write these equations as:

$$\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^{\mathbf{T}}\mathbf{P} - \mathbf{PBR}\,\mathbf{B}^{\mathbf{T}}\mathbf{P} = \mathbf{0} \qquad (4.1)$$

Where:

$\mathbf{R} = \left(\mathbf{R}_1^{-1} - \mathbf{R}_2^{-1}\right)$

Note that: for $\mathbf{R_1 = 2r_1 I}$, $\mathbf{R_2 = 2r_2 I}$, $\mathbf{R = 2\left(\dfrac{r_1 r_2}{r_2 - r_1}\right) I}$; and for $\mathbf{R}$ to be positive definite $\mathbf{r_2 > r_1}$. In fact if $\mathbf{r_2 >> r_1} \Rightarrow \mathbf{R = 2r_1\, I}$.

The approach adopted to solve the MRDE involves an inverse matrix technique, where the solution an inverse matrix version of the MRDE is first obtained and then by re-inverting the resulting solution, the Riccati matrix $[P]$ is obtained. This approach is given in Appendix A.3. Equation (4.1) is solved in Appendix-3 in some detail to obtain an expression for $\mathbf{E}$ and then re-inverted to obtain expressions for elements of $\mathbf{P}$. For the particular case considered in Section.3, (where: $\mathbf{s_{11} = s_{22} = s_{33} = 2s_1}$; $\mathbf{s_{14} = s_{25} = s_{36} = s_2}$; $\mathbf{s_{44} = s_{55} = s_{66} = 2s_3}$ and $\mathbf{r_{11} = r_{22} = r_{33} = 2r}$, $\mathbf{T = t_f - t}$) we get:

$$\mathbf{p_{11} = p_{22} = p_{33} = \frac{24r\left[4rs_1 + \left(4s_1s_3 - s_2^2\right)T\right]}{\left[48r^2 + 48rs_3\,T + 24rs_2\,T^2 + 16rs_1\,T^3 + \left(4s_1s_3 - s_2^2\right)T^4\right]}} \tag{4.2}$$

$$\mathbf{p_{44} = p_{55} = p_{66} = \frac{8r\left[12rs_3 + 12rs_2T + 12rs_1T^2 + \left(4s_1s_3 - s_2^2\right)T^3\right]}{\left[48r^2 + 48rs_3\,T + 24rs_2\,T^2 + 16rs_1\,T^3 + \left(4s_1s_3 - s_2^2\right)T^4\right]}} \tag{4.3}$$

$$\mathbf{p_{14} = p_{25} = p_{26} = \frac{12r\left[4rs_2 + 8rs_1T + \left(4s_1s_3 - s_2^2\right)T^2\right]}{\left[48r^2 + 48rs_3\,T + 24rs_2\,T^2 + 16rs_1\,T^3 + \left(4s_1s_3 - s_2^2\right)T^4\right]}} \tag{4.4}$$

4.1.1 Pursuer/Evader State Feedback Guidance Gains

The feedback gain matrix for the interceptor is given by:

$$\mathbf{K_1^p = \frac{1}{2r_1} B^T P = \frac{1}{2r_1}\begin{bmatrix} p_{14} & 0 & 0 & p_{44} & 0 & 0 \\ 0 & p_{25} & 0 & 0 & p_{55} & 0 \\ 0 & 0 & p_{36} & 0 & 0 & p_{66} \end{bmatrix}} \tag{4.5}$$

The feedback gain matrix for the target is given by:

$$\mathbf{K_2^e = \frac{1}{2r_2} B^T P = \frac{1}{2r_2}\begin{bmatrix} p_{14} & 0 & 0 & p_{44} & 0 & 0 \\ 0 & p_{25} & 0 & 0 & p_{55} & 0 \\ 0 & 0 & p_{36} & 0 & 0 & p_{66} \end{bmatrix}} \tag{4.6}$$

4.1.1.1 A Special Case

Another case of interest is when: $\mathbf{s_{11} = s_{22} = s_{33} = 2s_1}$; $\mathbf{s_{14} = s_{25} = s_{36} = s_2 = 0}$; $\mathbf{s_{44} = s_{55} = s_{66} = 2s_3 = 0}$ and $\mathbf{r_{11} = r_{22} = r_{33} = 2r}$. This is equivalent to setting the weightings on the velocity terms in the performance index to zero; in which case the latter is only a function of the miss distance squared and terms in interceptor and target accelerations. For this case:

$$\mathbf{p_{11} = p_{22} = p_{33} = \frac{6rs_1}{\left[3r + s_1\,T^3\right]}} \tag{4.7}$$

$$p_{14} = p_{25} = p_{36} = \frac{6rs_1 T}{\left[3r + s_1\ T^3\right]} \tag{4.8}$$

$$p_{44} = p_{55} = p_{66} = \frac{6rs_1 T^2}{\left[3r + s_1\ T^3\right]} \tag{4.9}$$

➔

$$\begin{bmatrix} p_{44} \\ p_{55} \\ p_{66} \end{bmatrix} = T \begin{bmatrix} p_{14} \\ p_{25} \\ p_{36} \end{bmatrix} = T^2 \begin{bmatrix} p_{11} \\ p_{22} \\ p_{33} \end{bmatrix} \tag{4.10}$$

Substituting for p_{ij} from (4.8) – (4.9), the feedback gain matrix for the pursuer in this case is given by:

$$K_1^p = \frac{1}{2r_1} B^T P = \frac{3rs_1 T}{r_1\left[3r + s_1\ T^3\right]} \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \end{bmatrix} \tag{4.11}$$

The feedback gain matrix for the evader is given by:

$$K_2^e = \frac{1}{2r_2} B^T P = \frac{3rs_1 T}{r_2\left[3r + s_1\ T^3\right]} \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \end{bmatrix} \tag{4.12}$$

4.2 Solution of the Vector Riccati Differential Equations (VRDE)

The VRDE given in equation (3.18) may be written as:

$$\dot{\underline{\xi}} = -\left[A - BR^{-1}B^T P\right]^T \underline{\xi} - PB\left(\underline{a}_1^d - \underline{a}_2^d\right) \tag{4.13}$$

Writing: $\underline{\xi} = \left[\xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4 \quad \xi_5 \quad \xi_6\right]^T$; equation (4.13) (in its decomposed form) may be written as (see Appendix A.4)):

$$\dot{\xi}_1 = \frac{p_{14}}{r_{11}} \xi_4 - p_{14}\left(a_{x_1}^d - a_{x_2}^d\right) \tag{4.14}$$

$$\dot{\xi}_2 = \frac{p_{25}}{r_{22}} \xi_5 - p_{25}\left(a_{y_1}^d - a_{y_2}^d\right) \tag{4.15}$$

$$\dot{\xi}_3 = \frac{p_{36}}{r_{33}} \xi_6 - p_{36}\left(a_{z_1}^d - a_{z_2}^d\right) \tag{4.16}$$

$$\dot{\xi}_4 = -\xi_1 + \frac{p_{44}}{r_{11}} \xi_4 - p_{44}\left(a_{x_1}^d - a_{x_2}^d\right) \tag{4.17}$$

$$\dot{\xi}_5 = -\xi_2 + \frac{p_{55}}{r_{22}} \xi_5 - p_{55}\left(a_{y_1}^d - a_{y_2}^d\right) \tag{4.18}$$

$$\dot{\xi}_6 = -\xi_3 + \frac{p_{66}}{r_{33}} \xi_6 - p_{66}\left(a_{z_1}^d - a_{z_2}^d\right) \tag{4.19}$$

Unfortunately it is not easily possible to obtain analytical solutions to equations (4.14) – (4.19), except for special cases where: $\left(a_{x_i}^d, a_{x_i}^d, a_{x_i}^d\right)$, $\left(a_{y_i}^d, a_{y_i}^d, a_{z_i}^d\right)$ and $\left(a_{z_i}^d, a_{z_i}^d, a_{z_i}^d\right)$,

$\mathbf{i=1,2}$ are constants. This case will be considered later on in this section. In general, however, equations (4.14) – (4.19) have to be solved backwards in time. For this purpose we make the substitutions:

Let $\mathbf{T=t_f-t}$, $\Rightarrow$ $\mathbf{dT=-dt}$; $\underline{\xi}(\mathbf{t})=\underline{\xi}(\mathbf{t_f-T})=\underline{\eta}(\mathbf{T})$; $\mathbf{a^d_{\gamma_i}(t)=a^d_{\gamma_i}(t_f-T)=\alpha^d_{\gamma_i}(T)}$; $\mathbf{i=1,2}$ $\gamma=\mathbf{x,y,z}$.

Hence the above equations (4.14) – (4.19) may be written as:

$$-\frac{d\eta_1}{dT}=\frac{p_{14}}{r_{11}}\eta_4-p_{14}\left(\alpha^d_{x_1}-\alpha^d_{x_2}\right) \tag{4.20}$$

$$-\frac{d\eta_2}{dT}=\frac{p_{25}}{r_{22}}\eta_5-p_{25}\left(\alpha^d_{y_1}-\alpha^d_{y_2}\right) \tag{4.21}$$

$$-\frac{d\eta_3}{dT}=\frac{p_{36}}{r_{33}}\eta_6-p_{36}\left(\alpha^d_{z_1}-\alpha^d_{z_2}\right) \tag{4.22}$$

$$-\frac{d\eta_4}{dT}=-\eta_1+\frac{p_{44}}{r_{11}}\eta_4-p_{44}\left(\alpha^d_{x_1}-\alpha^d_{x_2}\right) \tag{4.23}$$

$$-\frac{d\eta_5}{dT}=-\eta_2+\frac{p_{55}}{r_{22}}\eta_5-p_{55}\left(\alpha^d_{y_1}-\alpha^d_{y_2}\right) \tag{4.24}$$

$$-\frac{d\eta_6}{dT}=-\eta_3+\frac{p_{66}}{r_{33}}\eta_6-p_{66}\left(\alpha^d_{z_1}-\alpha^d_{z_2}\right) \tag{4.25}$$

These equations satisfy the boundary condition that $\underline{\eta}(\mathbf{0})=\underline{\xi}(\mathbf{t_f})=\underline{\mathbf{0}}$, and must be solved backwards in time, that is, $\mathbf{T}\to\mathbf{0}$. In the absence of an explicit closed form solution to equations (4.20) – (4.25) we shall write the general solution as:

$$\underline{\eta}(T)=\underline{\psi}\left[\,T,p_{ij},\left(\alpha^d_{x_1}-\alpha^d_{x_2}\right),\left(\alpha^d_{y_1}-\alpha^d_{y_2}\right),\left(\alpha^d_{z_1}-\alpha^d_{z_2}\right)\,\right] \tag{4.26}$$

4.2.1 Analytical Solution of the VRDE

Analytical solution of the VRDE is possible for the case when: $\mathbf{s_{11}=s_{22}=s_{33}=2s_1}$; $\mathbf{s_{14}=s_{25}=s_{36}=s_2=0}$; $\mathbf{s_{44}=s_{55}=s_{66}=2s_3=0}$ and $\mathbf{r_{11}=r_{22}=r_{33}=2r}$; $\alpha_{x_i},\alpha_{y_i},\alpha_{z_i}$ are constants; equations. For this case (see Appendix-A.4), the solution of equations (4.20) – (4.25) gives us:

$$\eta_1=\left[\frac{3rs_1T^2}{3r+s_1T^3}\right]\left(\alpha^d_{x_1}-\alpha^d_{x_2}\right) \tag{4.27}$$

$$\eta_2=\left[\frac{3rs_1T^2}{3r+s_1T^3}\right]\left(\alpha^d_{y_1}-\alpha^d_{y_2}\right) \tag{4.28}$$

$$\eta_3=\left[\frac{3rs_1T^2}{3r+s_1T^3}\right]\left(\alpha^d_{z_1}-\alpha^d_{z_2}\right) \tag{4.29}$$

$$\eta_4=\left[\frac{3rs_1T^3}{3r+s_1T^3}\right]\left(\alpha^d_{x_1}-\alpha^d_{x_2}\right) \tag{4.30}$$

$$\eta_5=\left[\frac{3rs_1T^3}{3r+s_1T^3}\right]\left(\alpha^d_{y_1}-\alpha^d_{y_2}\right) \tag{4.31}$$

$$\eta_6 = \left[\frac{3rs_1T^3}{3r + s_1T^3}\right]\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \tag{4.31}$$

Noting that: $\underline{\eta}(T) = \underline{\xi}(t_f - T)$, the feedback for the disturbance term may be written as:

$$K^d_1\underline{\eta} = R_1^{-1}B^T\underline{\eta} = \frac{1}{2r_1}\begin{bmatrix}\eta_4\\ \eta_5\\ \eta_6\end{bmatrix} = \frac{1}{2r_1}\left[\frac{3rs_1T^3}{3r+2s_1T^3}\right]\begin{bmatrix}\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right)\\ \left(\alpha^d_{y_1} - \alpha^d_{y_2}\right)\\ \left(\alpha^d_{z_1} - \alpha^d_{z_2}\right)\end{bmatrix} \tag{4.32}$$

$$K^d_2\underline{\eta} = R_2^{-1}B^T\underline{\eta} = \frac{1}{2r_2}\begin{bmatrix}\eta_4\\ \eta_5\\ \eta_6\end{bmatrix} = \frac{1}{2r_2}\left[\frac{3rs_1T^3}{3r+2s_1T^3}\right]\begin{bmatrix}\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right)\\ \left(\alpha^d_{y_1} - \alpha^d_{y_2}\right)\\ \left(\alpha^d_{z_1} - \alpha^d_{z_2}\right)\end{bmatrix} \tag{4.33}$$

4.3 Mechanisation of the Game Theoretic Guidance

Using the expressions for the optimum guidance equations (4.32) and (4.33), and those for the feed-back gains given in equations (4.32) and (4.33), we get, for the general case: $s_{11} = s_{22} = s_{33} = 2s_1$; $s_{14} = s_{25} = s_{36} = s_2$; $s_{44} = s_{55} = s_{66} = 2s_3$ and $r_{11} = r_{22} = r_{33} = 2r$:

$$\underline{a}^p_1 = -K^p_1\underline{y}_{12} - K^d_1\underline{\eta} = -\frac{1}{2r_1}\begin{bmatrix}p_{14}x_{12} + p_{44}u_{12} + \eta_4\\ p_{25}y_{12} + p_{55}v_{12} + \eta_5\\ p_{36}z_{12} + p_{66}w_{12} + \eta_6\end{bmatrix} \tag{4.34}$$

$$\underline{a}^e_2 = -K^e_2\underline{y}_{12} - K^d_2\underline{\eta} = -\frac{1}{2r_2}\begin{bmatrix}p_{14}x_{12} + p_{44}u_{12} + \eta_4\\ p_{25}y_{12} + p_{55}v_{12} + \eta_5\\ p_{36}z_{12} + p_{66}w_{12} + \eta_6\end{bmatrix} \tag{4.35}$$

This is depicted as a network block diagram Figure 4-1. For the case when: $s_{11} = s_{22} = s_{33} = 2s_1$; $s_{14} = s_{25} = s_{36} = s_2 = 0$; $s_{44} = s_{55} = s_{66} = 2s_3 = 0$ and $r_{11} = r_{22} = r_{33} = 2r$; $\alpha_{x_i}, \alpha_{y_i}, \alpha_{z_i}$ constants, we get:

$$\underline{a}^p_1 = -K^p_1\underline{y}_{12} - K^d_1\underline{\eta} = \frac{-3rs_1T}{2r_1\left[3r + 2s_1\,T^3\right]}\begin{bmatrix}x_{12} + Tu_{12} + T^2\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right)\\ y_{12} + Tv_{12} + T^2\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right)\\ z_{12} + Tw_{12} + T^2\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right)\end{bmatrix} \tag{4.36}$$

$$\underline{a}_2^e = -K_2^e \underline{y}_{12} - K_2^d \underline{\eta} = \frac{-3rs_1T}{2r_2\left[3r + 2s_1\ T^3\right]} \begin{bmatrix} x_{12} + Tu_{12} + T^2\left(\alpha_{x_1}^d - \alpha_{x_2}^d\right) \\ y_{12} + Tv_{12} + T^2\left(\alpha_{y_1}^d - \alpha_{y_2}^d\right) \\ z_{12} + Tw_{12} + T^2\left(\alpha_{z_1}^d - \alpha_{z_2}^d\right) \end{bmatrix} \tag{4.37}$$

5. Extension to Optimum Guidance

The development of the optimum guidance not involving target evasion manoeuvre can proceed directly from the game theory development presented in earlier sections. For this case, the objective function takes the following form:

$$V\left(\underline{x}_{12}, \underline{u}_{12}, \underline{a}_1^p, \underline{a}_2^e\right) = \left\{ s_1 \left\|\underline{x}_{12}\right\|^2 \right\}_{t=t_f} + \int_0^{t_f} \left\{ + r_1 \left\|\underline{a}_1^p\right\|^2 \right\} dt \tag{5.1}$$

Where:

$S = \begin{bmatrix} 2s_1 I & 0 \\ 0 & 0 \end{bmatrix}$**:** is a 6x6 final state function penalty weighting matrix. $\mathbf{I}$**:** is a 3x3 unity matrix.

$R_1 = r_1 I$**:** is a 3x3 pursuer's demanded acceleration function penalty weighting matrix.

Note that in this case, the first term in the objective function is only the miss distance and that the target game theory based evasion manoeuvre is not present. Similarly in the kinematics equations pre-specified target acceleration $\underline{a}_2^e$ is present whereas $\underline{a}_1^d$ is zero. The kinematics equation for the engagement, equation (3.3) may be written as:

$$\frac{d}{dt}\underline{y}_{12} = A\,\underline{y}_{12} + B\,\underline{a}_1^p - B\,\underline{a}_2^d \tag{5.2}$$

Following the approach presented earlier it follows that the optimum guidance law for the interceptor in this case is given by:

$$\underline{a}_1^p = -R_1^{-1}B^TP\,\underline{y}_{12} - R_1^{-1}B^T\underline{\xi} = -K_1^p\ \underline{y}_{12} - K_1^d\,\underline{\xi} \tag{5.3}$$

Or in terms of time-to-go (5.3) has the form

$$\underline{a}_1^p = -R_1^{-1}B^TP\,\underline{y}_{12} - R_1^{-1}B^T\underline{\eta} = -K_1^p\ \underline{y}_{12} - K_1^d\,\underline{\eta} \tag{5.4}$$

Where the MRDE and VRDE are the same as those previously derived in section 4. That is:

$$\dot{P} + PA + A^TP - PB\,R_1^{-1}B^TP = 0 \tag{5.5}$$

And:

$$-\dot{\underline{\xi}} - \left\lfloor A - B\,R_1^{-1}B^TP \right\rfloor^T + PB\,\underline{a}_2^d = 0 \tag{5.6}$$

As far as the solution of these equations (5.5), (5.6) is concerned, these are identical to those derived earlier with $\mathbf{r}$ replaced by $\mathbf{r_1}$ and $\mathbf{s_2 = s_3 = 0}$ It can be easily verified that, for this case:

$$p_{11} = p_{22} = p_{33} = \frac{6r_1 s_1}{\left[3_1 r + s_1 T^3\right]} \tag{5.7}$$

$$p_{14} = p_{25} = p_{26} = \frac{6r_1 s_1 T}{\left[3r_1 + s_1 T^3\right]} \tag{5.8}$$

$$p_{44} = p_{55} = p_{66} = \frac{6r_1 s_1 T^2}{\left[3r_1 + s_1 T^3\right]} \tag{5.9}$$

The feedback gain matrix for the interceptor in this case is given by:

$$K_1^p = \frac{1}{2r_1} B^T P = \frac{3s_1 T}{\left[3r_1 + s_1 T^3\right]} \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \end{bmatrix} \tag{5.10}$$

And the disturbance term for constant target manoeuvre $\underline{a}_2^d$ is given by:

$$K_1^d \underline{\eta} = R_1^{-1} B^T \underline{\eta} = \frac{1}{2r_1} \begin{bmatrix} \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = -\frac{1}{2}\left[\frac{3s_1 T^3}{3r_1 + 2s_1 T^3}\right] \begin{bmatrix} \alpha_{x2}^d \\ \alpha_{y2}^d \\ \alpha_{z2}^d \end{bmatrix} \tag{5.11}$$

Remarks
It is interesting to note that if $\mathbf{r_1 \to 0}$, then (5.11), (5.12) become:

$$K_1^p = \frac{1}{2r_1} B^T P = \frac{3}{T^2} \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \end{bmatrix} \tag{5.12}$$

$$K_1^d \underline{\eta} = R_1^{-1} B^T \underline{\eta} = \frac{1}{2r_1} \begin{bmatrix} \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} \alpha_{x2}^d \\ \alpha_{y2}^d \\ \alpha_{z2}^d \end{bmatrix} \tag{5.13}$$

This relationship is the 'link' between the optimal guidance and the conventional guidance such as PN and APN, and will be further elaborated in the next section.

6. Relationship with the Proportional Navigation (PN) and the Augmented PN Guidance

In order to establish a connection between the optimal guidance and the PN and APN we shall assume that the engagement trajectory is such that the azimuth and elevation sightline angles ψ_{21}, θ_{21} (Figure A-1) remain small during engagement, that is, the

trajectory remains close to collision close geometry. For this condition it follows that the interceptor/target relative velocity is pointed approximately along the sight line and is approximately equal to the closing velocity $\mathbf{V_c}$. Using equations (5.10)- (5.13), the state feedback guidance acceleration $\underline{\mathbf{a}}_1^p$ we shall write as:

$$\underline{\mathbf{a}}_1^p = -\mathbf{K}_1^p \underline{\mathbf{y}}_{12} - \mathbf{K}_1^p \underline{\eta} = -3\mathbf{V_c} \begin{bmatrix} \frac{1}{V_c T^2} x_{12} + \frac{1}{V_c T} u_{12} \\ \frac{1}{V_c T^2} y_{12} + \frac{1}{V_c T} v_{12} \\ V_c \frac{1}{T^2} z_{12} + \frac{1}{V_c T} w_{12} \end{bmatrix} + \frac{3}{2} \begin{bmatrix} \alpha_{x_2}^d \\ \alpha_{y_2}^d \\ \alpha_{z_2}^d \end{bmatrix} \tag{6.1}$$

It is shown in Appendix A5, equations (A5-3)-(A5-5) that:

$\dot{\psi}_{21} = -\left(\frac{v_{12}}{V_c T} + \frac{y_{12}}{V_c T^2} \right)$; $\dot{\theta}_{21} = -\left(\frac{w_{12}}{V_c T} + \frac{z_{12}}{V_c T^2} \right)$; and $\frac{1}{V_c T^2} x_{12} + \frac{1}{V_c T} u_{12} = 0$

Thus equation (6.1) reduces to:

$$\underline{\mathbf{a}}_1^p = 3\mathbf{V_c} \begin{bmatrix} 0 \\ \dot{\psi}_{21} \\ \dot{\theta}_{21} \end{bmatrix} + \frac{3}{2} \begin{bmatrix} \alpha_{x_2}^d \\ \alpha_{y_2}^d \\ \alpha_{z_2}^d \end{bmatrix} \tag{6.2}$$

This is the well-known augmented proportional guidance (APN) law when the target manoeuvre is constant. The navigation gain is $\mathbf{3V_c}$ associated with the sightline rate, and $\frac{3}{2}$ associated with target acceleration. When there is no target manoeuvre we get the proportional navigation (PN) guidance.

7. Conclusions

This report was focused on the application of GTG where the interceptor's (pursuer's) objective was to hit the target while the target's (evader's) strategy was to avoid intercept. Since the parties involved in the engagement derive their respective strategies based on the assessment of the current relative states (i.e. the environment) and optimisation of the objective function (i.e. the decision making criteria), we shall term this type of guidance as 'intelligent' guidance. It was further shown that OG was a special case of the GTG. Both these guidance techniques follow similar procedures for deriving the state feedback guidance algorithm; the objective function (performance index), however, in the two cases were different. A 3-D interceptor/target engagement was considered and the various guidance laws were derived that will allow design engineers the flexibility to choose guidance gains (by selecting appropriate performance index weights) to meet their specific engagement objectives. For a number of important cases closed form expressions

have been obtained for the feedback gains. A relationship between the GTG, the OG and the classical PN and APN has also been demonstrated.

The game theory based guidance technique proposed in this report may provide a useful tool to study vulnerabilities of existing weapons systems against current and future threat systems that may incorporate 'intelligent' guidance or for enhancing the capability of future weapons by implementing (game theory based) intelligent guidance. Further research is required in this area in order to evaluate the performance of the game theoretic guidance in realistic combat environments. Further research is required to test the guidance technique proposed for: imperfect measurement of the guidance seeker, system lags, and robustness to variations in parameters and noise.

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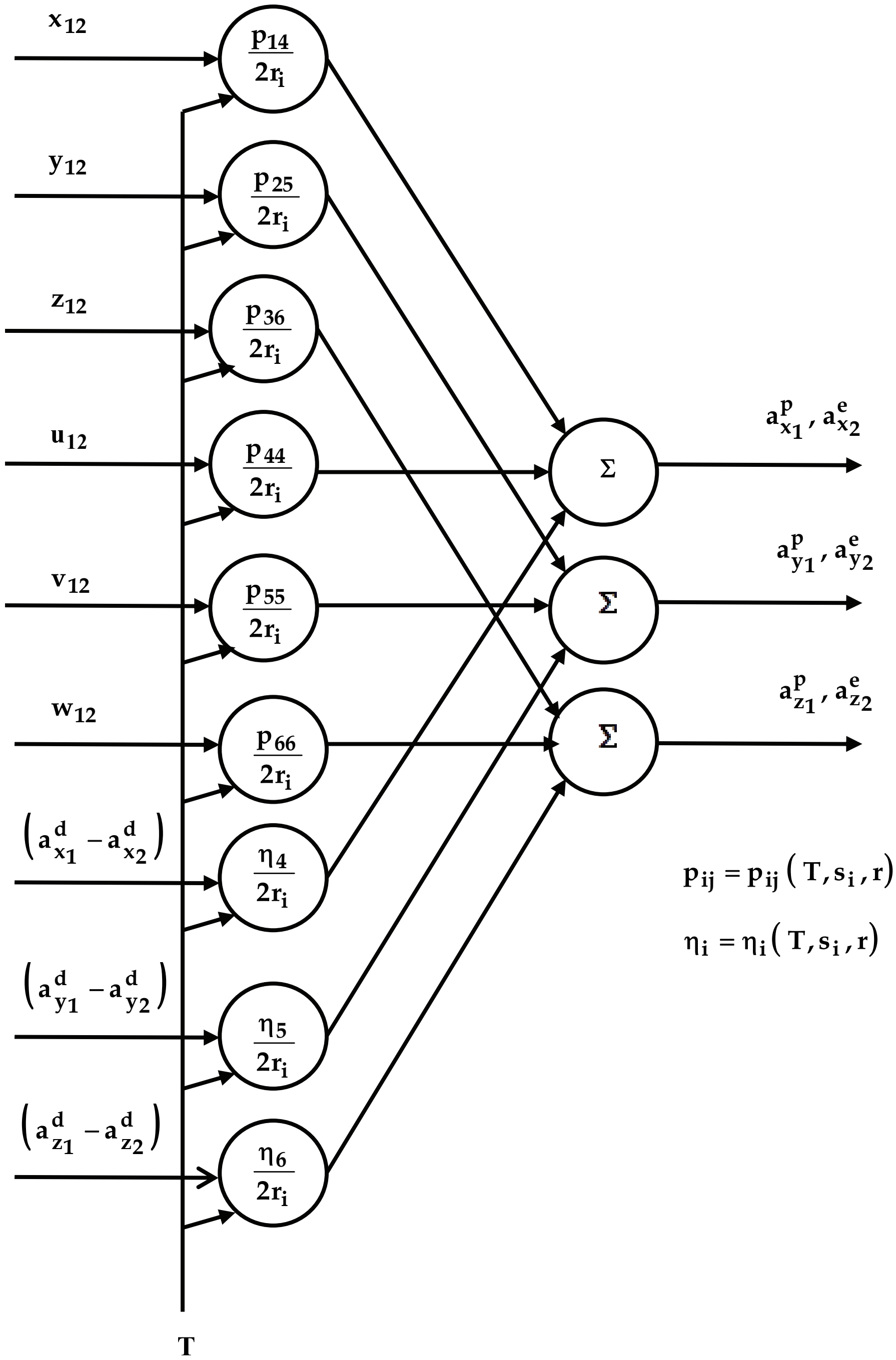

Figure 5-1. Network Block diagram of the guidance mechanisation

Appendix A

A.1 Verifying the Positive Semi-Definiteness of Matrix S

Now, the determinant of the block matrix $\mathbf{S}$ may be written as:

$$|\mathbf{S}| = \begin{vmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} \\ \mathbf{S_{12}} & \mathbf{S_{22}} \end{vmatrix} = |\mathbf{S_{11}}|\left|\mathbf{S_{22}} - \mathbf{S_{12}}^T \mathbf{S_{11}}^{-1} \mathbf{S_{12}}\right| \tag{A1-1}$$

For $\mathbf{S}$ to be positive semi-definite, we must have:

$$|\mathbf{S}| = \begin{vmatrix} 2s_1\mathbf{I} & s_2\mathbf{I} \\ s_2\mathbf{I} & 2s_3\mathbf{I} \end{vmatrix} = 2s_1\left|2s_3\mathbf{I} - \frac{s_2{}^2}{2s_1}\mathbf{I}\right| = 2s_1\left(2s_3 - \frac{s_2{}^2}{2s_1}\right) \geq 0 \tag{A1-2}$$

That is: $s_1 \geq 0$, and $\left(2s_3 - \frac{s_2{}^2}{2s_1}\right) \geq 0$ or $4s_1s_3 \geq s_2{}^2$ (A1-3)

Note that: s_2 can be either positive or negative.

A.2 Derivation of Riccati Differential Equations

Substituting for $\mathbf{a}_1^p$ and $\mathbf{a}_2^e$ from equations (3.13) into equation (3.3) gives us:

$$\begin{aligned} \frac{d}{dt}\underline{\mathbf{y}}_{12} = \mathbf{A}\underline{\mathbf{y}}_{12} - \mathbf{B}\left(\mathbf{R}_1{}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{R}_1{}^{-1}\mathbf{B}^T\underline{\xi} - \underline{\mathbf{a}}_1^d\right)\cdots \\ + \mathbf{B}\left(\mathbf{R}_2{}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{R}_2{}^{-1}\mathbf{B}^T\underline{\xi} - \underline{\mathbf{a}}_2^d\right) \end{aligned} \tag{A2-1}$$

Equation (3.15) may be written as:

$$\mathbf{A}^T\left(\mathbf{P}\underline{\mathbf{y}}_{12} + \underline{\xi}\right) = -\dot{\underline{\lambda}} = -\dot{\mathbf{P}}\,\underline{\mathbf{y}}_{12} - \mathbf{P}\,\dot{\underline{\mathbf{y}}}_{12} - \dot{\underline{\xi}}$$

Substituting for $\dot{\underline{\mathbf{y}}}_{12}$ from equation (A2-1) gives us:

$$\begin{aligned} \mathbf{A}^T\left(\mathbf{P}\underline{\mathbf{y}}_{12} + \underline{\xi}\right) = -\dot{\mathbf{P}}\underline{\mathbf{y}}_{12} - \mathbf{P}\Big\{\mathbf{A}\underline{\mathbf{y}}_{12} - \mathbf{B}\left(\mathbf{R}_1{}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{R}_1{}^{-1}\mathbf{B}^T\underline{\xi} - \underline{\mathbf{a}}_1^d\right)\cdots \\ + \mathbf{B}\left(\mathbf{R}_2{}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{R}_2{}^{-1}\mathbf{B}^T\underline{\xi} - \underline{\mathbf{a}}_2^d\right)\Big\} - \dot{\underline{\xi}} \end{aligned}$$

➔

$$\begin{aligned} \mathbf{A}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{A}^T\underline{\xi} = -\dot{\mathbf{P}}\underline{\mathbf{y}}_{12} - \mathbf{P}\mathbf{A}\underline{\mathbf{y}}_{12} + \mathbf{P}\mathbf{B}\left(\mathbf{R}_1{}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{R}_1{}^{-1}\mathbf{B}^T\underline{\xi} - \underline{\mathbf{a}}_1^d\right)\cdots \\ - \mathbf{P}\mathbf{B}\left(\mathbf{R}_2{}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{y}}_{12} + \mathbf{R}_2{}^{-1}\mathbf{B}^T\underline{\xi} - \underline{\mathbf{a}}_2^d\right) - \dot{\underline{\xi}} \end{aligned} \tag{A2-2}$$

Re-arranging the terms, equation (A2-2) can be written as:

$$\begin{aligned} \left[\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\left(\mathbf{R}_1{}^{-1} - \mathbf{R}_2{}^{-1}\right)\mathbf{B}^T\mathbf{P}\right]\ \underline{\mathbf{y}}_{12}\cdots \\ = -\dot{\underline{\xi}} - \left[\mathbf{A}^T - \mathbf{P}\mathbf{B}\left(\mathbf{R}_1{}^{-1} - \mathbf{R}_2{}^{-1}\right)\mathbf{B}^T\right]\underline{\xi} - \mathbf{P}\mathbf{B}\left(\underline{\mathbf{a}}_1^d - \underline{\mathbf{a}}_2^d\right) \end{aligned} \tag{A2-3}$$

A solution of equation (A2-3) is obtained if the LHS and the RHS are both equal to zero; that is:

$$\left[\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^{\mathbf{T}}\mathbf{P} - \mathbf{PB}\left(\mathbf{R_1}^{-1} - \mathbf{R_2}^{-1}\right)\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]\ \underline{\mathbf{y}}_{12} = 0$$

And

$$-\dot{\underline{\xi}} - \left[\mathbf{A}^{\mathbf{T}} - \mathbf{PB}\left(\mathbf{R_1}^{-1} - \mathbf{R_2}^{-1}\right)\mathbf{B}^{\mathbf{T}}\right]\ \underline{\xi} - \mathbf{PB}\left(\underline{\mathbf{a}}_1^{\mathbf{d}} - \underline{\mathbf{a}}_2^{\mathbf{d}}\right) = 0$$

Since the above equations hold for all $\underline{\mathbf{y}}_{12}$, $\underline{\mathbf{a}}_1^{\mathbf{d}}$ and $\underline{\mathbf{a}}_2^{\mathbf{d}}$; we get:

$$\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^{\mathbf{T}}\mathbf{P} - \mathbf{PB}\left(\mathbf{R_1}^{-1} - \mathbf{R_2}^{-1}\right)\mathbf{B}^{\mathbf{T}}\mathbf{P} = 0 \tag{A2-4}$$

$$-\dot{\underline{\xi}} - \left[\mathbf{A}^{\mathbf{T}} - \mathbf{PB}\left(\mathbf{R_1}^{-1} - \mathbf{R_2}^{-1}\right)\mathbf{B}^{\mathbf{T}}\right]\ \underline{\xi} - \mathbf{PB}\left(\underline{\mathbf{a}}_1^{\mathbf{d}} - \underline{\mathbf{a}}_2^{\mathbf{d}}\right) = 0 \tag{A2-5}$$

These differential equations satisfy the boundary conditions $\mathbf{P}(\mathbf{t_f}) = \mathbf{S}$ and $\underline{\xi}(\mathbf{t_f}) = \mathbf{0}$. In the sequel, equation (A2-4) will be referred to as the Matrix Riccati Differential Equation (MRDE) and equation (A2-5) will be referred to as the Vector Riccati Differential Equation (VRDE). For convenience we shall write: $\mathbf{R}^{-1} = \left(\mathbf{R_1}^{-1} - \mathbf{R_2}^{-1}\right)$.

A.3 Solving the Matrix Riccati Differential Equation

Let us write the $\mathbf{P}$ matrix as:

$$\mathbf{P} = \mathbf{E}^{-1};\ \Rightarrow\ \mathbf{PE} = \mathbf{I};\ \Rightarrow\ \dot{\mathbf{P}}\mathbf{E} + \mathbf{P}\dot{\mathbf{E}} = 0;\ \Rightarrow\ \dot{\mathbf{P}} = -\mathbf{E}^{-1}\dot{\mathbf{E}}\,\mathbf{E}^{-1} \tag{A3-1}$$

Substituting for $\mathbf{P}$ in equation (A2-4, we obtain the inverse-MRDE for $\mathbf{E}$ as:

$$\dot{\mathbf{E}} = \mathbf{AE} + \mathbf{EA}^{\mathbf{T}} - \mathbf{BR}^{-1}\mathbf{B}^{\mathbf{T}} \tag{A3-2}$$

We shall solve for $\mathbf{E}$, the inverse Riccati matrix first, and then invert this to obtain the Riccati solution $\mathbf{P}$. Because both $\mathbf{P}$ and $\mathbf{E}$ matrices are symmetric; we may write:

$$\mathbf{E} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{12} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{13} & e_{23} & e_{33} & e_{34} & e_{35} & e_{36} \\ e_{14} & e_{24} & e_{34} & e_{44} & e_{45} & e_{46} \\ e_{15} & e_{25} & e_{35} & e_{45} & e_{55} & e_{56} \\ e_{16} & e_{26} & e_{36} & e_{46} & e_{56} & e_{66} \end{bmatrix} \tag{A3-3}$$

The terminal condition for matrix $\mathbf{E}(\mathbf{t_f})$ is given by: $\mathbf{E}(\mathbf{t_f}) = \mathbf{S}^{-1}$. Let us, therefore define:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} \\ \mathbf{S_{12}} & \mathbf{S_{22}} \end{bmatrix} = \begin{bmatrix} s_{11} & 0 & 0 & s_{14} & 0 & 0 \\ 0 & s_{22} & 0 & 0 & s_{25} & 0 \\ 0 & 0 & s_{33} & 0 & 0 & s_{36} \\ s_{14} & 0 & 0 & s_{44} & 0 & 0 \\ 0 & s_{25} & 0 & 0 & s_{55} & 0 \\ 0 & 0 & s_{36} & 0 & 0 & s_{66} \end{bmatrix} \tag{A3-4}$$

Where:

$$[S_{11}] = \begin{bmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix}; \ [S_{12}] = \begin{bmatrix} s_{14} & 0 & 0 \\ 0 & s_{25} & 0 \\ 0 & 0 & s_{36} \end{bmatrix}; \ [S_{22}] = \begin{bmatrix} s_{44} & 0 & 0 \\ 0 & s_{55} & 0 \\ 0 & 0 & s_{66} \end{bmatrix}$$

For the particular case considered in section 3.1, $s_{11} = s_{22} = s_{33} = 2s_1$; $s_{14} = s_{25} = s_{36} = s_2$, and $s_{44} = s_{55} = s_{66} = 2s_3$. We shall first consider the general matrix $\mathbf{S}$ as shown in equation and later substitute the particular values. We write the inverse of the inverse matrix $\mathbf{S}^{-1}$ as:

$$\mathbf{S}^{-1} = \mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{12} & \mathbf{T}_{22} \end{bmatrix} \quad \text{(A3-5)}$$

Note that, since $\mathbf{S}$ is symmetric then so is $\mathbf{T} = \mathbf{S}^{-1}$ and for $\mathbf{T}$ to be the inverse of $\mathbf{S}$, we must have:

$$\mathbf{S}\mathbf{S}^{-1} = \mathbf{S}\mathbf{T} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{12} & \mathbf{T}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

➔

$$\begin{bmatrix} (\mathbf{S}_{11}\mathbf{T}_{11} + \mathbf{S}_{12}\mathbf{T}_{12}) & (\mathbf{S}_{11}\mathbf{T}_{12} + \mathbf{S}_{12}\mathbf{T}_{22}) \\ (\mathbf{S}_{12}\mathbf{T}_{11} + \mathbf{S}_{22}\mathbf{T}_{12}) & (\mathbf{S}_{12}\mathbf{T}_{12} + \mathbf{S}_{22}\mathbf{T}_{22}) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

➔

$$(\mathbf{S}_{11}\mathbf{T}_{11} + \mathbf{S}_{12}\mathbf{T}_{12}) = \mathbf{I} \quad \text{(A3-6)}$$

$$(\mathbf{S}_{11}\mathbf{T}_{12} + \mathbf{S}_{12}\mathbf{T}_{22}) = \mathbf{0} \quad \text{(A3-7)}$$

$$(\mathbf{S}_{12}\mathbf{T}_{11} + \mathbf{S}_{22}\mathbf{T}_{12}) = \mathbf{0} \quad \text{(A3-8)}$$

$$(\mathbf{S}_{12}\mathbf{T}_{12} + \mathbf{S}_{22}\mathbf{T}_{22}) = \mathbf{I} \quad \text{(A3-9)}$$

Equation (A3-8) ➔

$$\mathbf{T}_{11} = -\mathbf{S}_{12}^{-1}\mathbf{S}_{22}\mathbf{T}_{12} \quad \text{(A3-10)}$$

Substituting equations (A3-10) into equation (A3-6) gives us:

$$\mathbf{T}_{12} = \left[\mathbf{S}_{12} - \mathbf{S}_{11}\mathbf{S}_{12}^{-1}\mathbf{S}_{22}\right]^{-1} \quad \text{(A3-11)}$$

Substituting equation (A3-11) into equation (A3-10) gives us:

$$\mathbf{T}_{11} = -\mathbf{S}_{12}^{-1}\mathbf{S}_{22}\left[\mathbf{S}_{12} - \mathbf{S}_{11}\mathbf{S}_{12}^{-1}\mathbf{S}_{22}\right]^{-1} \quad \text{(A3-12)}$$

Also, equation (A3-7) gives us:

$$\mathbf{T}_{22} = -\mathbf{S}_{12}^{-1}\mathbf{S}_{11}\mathbf{T}_{12} \quad \text{(A3-13)}$$

Equations (A3-11) and (A3-13) give us:

$$\mathbf{T}_{22} = -\mathbf{S}_{12}^{-1}\mathbf{S}_{11}\left[\mathbf{S}_{12} - \mathbf{S}_{11}\mathbf{S}_{12}^{-1}\mathbf{S}_{22}\right]^{-1} \quad \text{(A3-14)}$$

It follows from (A3-4) that:

$$\mathbf{S_{11}}^{-1} = \begin{bmatrix} \frac{1}{s_{11}} & 0 & 0 \\ 0 & \frac{1}{s_{22}} & 0 \\ 0 & 0 & \frac{1}{s_{33}} \end{bmatrix};\ \mathbf{S_{12}}^{-1} = \begin{bmatrix} \frac{1}{s_{14}} & 0 & 0 \\ 0 & \frac{1}{s_{25}} & 0 \\ 0 & 0 & \frac{1}{s_{36}} \end{bmatrix};\ \mathbf{S_{22}}^{-1} = \begin{bmatrix} \frac{1}{s_{44}} & 0 & 0 \\ 0 & \frac{1}{s_{55}} & 0 \\ 0 & 0 & \frac{1}{s_{66}} \end{bmatrix}$$

Using expressions for $\mathbf{S_{ij}}$ and $\mathbf{S_{ij}}^{-1}$, equations (A3-11), (A3-12) and (A3-14) give:

$$\mathbf{T_{11}} = \begin{bmatrix} \frac{s_{44}}{\left(s_{11}s_{44} - s_{14}{}^2\right)} & 0 & 0 \\ 0 & \frac{s_{55}}{\left(s_{22}s_{55} - s_{25}{}^2\right)} & 0 \\ 0 & 0 & \frac{s_{66}}{\left(s_{33}s_{66} - s_{36}{}^2\right)} \end{bmatrix} \tag{A3-15}$$

$$\mathbf{T_{12}} = \begin{bmatrix} \frac{-s_{14}}{\left(s_{11}s_{44} - s_{14}{}^2\right)} & 0 & 0 \\ 0 & \frac{-s_{25}}{\left(s_{22}s_{55} - s_{25}{}^2\right)} & 0 \\ 0 & 0 & \frac{-s_{36}}{\left(s_{33}s_{66} - s_{36}{}^2\right)} \end{bmatrix} \tag{A3-16}$$

$$\mathbf{T_{22}} = \begin{bmatrix} \frac{s_{11}}{\left(s_{11}s_{44} - s_{14}{}^2\right)} & 0 & 0 \\ 0 & \frac{s_{22}}{\left(s_{22}s_{55} - s_{25}{}^2\right)} & 0 \\ 0 & 0 & \frac{s_{33}}{\left(s_{33}s_{66} - s_{36}{}^2\right)} \end{bmatrix} \tag{A3-17}$$

$\mathbf{S}^{-1} = \mathbf{T} = \begin{bmatrix} \mathbf{T_{11}} & \mathbf{T_{12}} \\ \mathbf{T_{12}} & \mathbf{T_{22}} \end{bmatrix}$ matrix can now be constructed using expressions for $\mathbf{T_{11}, T_{12}, T_{22}}$.

Note that, when $s_{11} = s_{22} = s_{33} = 2s_1$; $s_{14} = s_{25} = s_{36} = s_2$, and $s_{44} = s_{55} = s_{66} = 2s_3$, then using equations (A3-15) – (A3-17) it can be shown that::

$$\mathbf{S}^{-1} = \frac{1}{\left(4s_1s_3 - s_2{}^2\right)} \begin{bmatrix} 2s_3\mathbf{I} & -s_2\mathbf{I} \\ -s_2\mathbf{I} & 2s_1\mathbf{I} \end{bmatrix}$$

(A3-18)

A.3.1. Solution of the Inverse Matrix Riccati Differential Equation

In this section we decompose equation (A3-2) in its elemental form to facilitate the solution of the inverse MRDE for $\mathbf{E}$.

Now:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \Rightarrow \ \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A3-19)}$$

And:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \Rightarrow \mathbf{B}^{\mathbf{T}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \text{and we write } \mathbf{R} = \begin{bmatrix} r_{11} & 0 & 0 \\ 0 & r_{11} & 0 \\ 0 & 0 & r_{11} \end{bmatrix}$$

It follows that from the above that:

$$\mathbf{AE} = \begin{bmatrix} e_{14} & e_{24} & e_{34} & e_{44} & e_{45} & e_{46} \\ e_{15} & e_{25} & e_{35} & e_{45} & e_{55} & e_{56} \\ e_{16} & e_{26} & e_{36} & e_{46} & e_{56} & e_{66} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \ \mathbf{EA}^{\mathbf{T}} = \begin{bmatrix} e_{14} & e_{15} & e_{16} & 0 & 0 & 0 \\ e_{24} & e_{25} & e_{26} & 0 & 0 & 0 \\ e_{34} & e_{35} & e_{36} & 0 & 0 & 0 \\ e_{44} & e_{45} & e_{46} & 0 & 0 & 0 \\ e_{45} & e_{55} & e_{56} & 0 & 0 & 0 \\ e_{46} & e_{56} & e_{66} & 0 & 0 & 0 \end{bmatrix} \qquad \text{(A3-20)}$$

Also:

$$\mathbf{B\,R^{-1}B^{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/r_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/r_{33} \end{bmatrix} \qquad \text{(A3-21)}$$

Hence the RHS of equation (A3-2) may be written as:

$$\mathbf{A\,E + E\,A^{T} - B\,R^{-1}B^{T}}$$
$$= \begin{bmatrix} 2e_{14} & (e_{15}+e_{24}) & (e_{16}+e_{34}) & e_{44} & e_{45} & e_{46} \\ (e_{15}+e_{24}) & 2e_{25} & (e_{26}+e_{35}) & e_{45} & e_{55} & e_{56} \\ (e_{16}+e_{34}) & (e_{26}+e_{35}) & 2e_{36} & e_{46} & e_{56} & e_{66} \\ e_{44} & e_{45} & e_{46} & -1/r_{11} & 0 & 0 \\ e_{45} & e_{55} & e_{56} & 0 & -1/r_{22} & 0 \\ e_{46} & e_{56} & e_{66} & 0 & 0 & -1/r_{33} \end{bmatrix} \qquad \text{(A3-22)}$$

Since $\mathbf{E}$ is a symmetric matrix, we need to consider only the elements of the upper triangular matrix. Thus, using equation (A3-22), differential equation (A3-2) may be written (in its elemental form) as:

$$\dot{e}_{11} = 2e_{14};\ \dot{e}_{12} = (e_{15} + e_{24});\ \dot{e}_{13} = (e_{16} + e_{34});\ \dot{e}_{14} = e_{44};\ \dot{e}_{15} = e_{45};\ \dot{e}_{16} = e_{46};$$

$$\dot{e}_{22} = 2e_{25};\ \dot{e}_{23} = (e_{26} + e_{35});\ \dot{e}_{24} = e_{45};\ \dot{e}_{25} = e_{55};\ \dot{e}_{26} = e_{56};$$

$$\dot{e}_{33} = 2e_{36};\ \dot{e}_{34} = e_{46};\ \dot{e}_{35} = e_{56};\ \dot{e}_{36} = e_{66};$$

$$\dot{e}_{44} = -1/r_{11};\ \dot{e}_{45} = 0;\ \dot{e}_{46} = 0$$

$$\dot{e}_{55} = -1/r_{22};\ \dot{e}_{56} = 0;$$

$$\dot{e}_{66} = -1/r_{33}$$

The terminal conditions are given by $\mathbf{E}(t_f) = \mathbf{S}^{-1}$ may be written as:

$$e_{11}(t_f) = \frac{s_{44}}{(s_{11}s_{44} - s_{14}^2)};\ e_{22}(t_f) = \frac{s_{55}}{(s_{22}s_{55} - s_{25}^2)};\ e_{33}(t_f) = \frac{s_{66}}{(s_{33}s_{66} - s_{36}^2)};$$

$$e_{14}(t_f) = \frac{-s_{14}}{(s_{11}s_{44} - s_{14}^2)};\ e_{25}(t_f) = \frac{-s_{25}}{(s_{22}s_{55} - s_{25}^2)};\ e_{36}(t_f) = \frac{-s_{36}}{(s_{33}s_{66} - s_{36}^2)};$$

$$e_{44}(t_f) = \frac{s_{11}}{(s_{11}s_{44} - s_{14}^2)};\ e_{55}(t_f) = \frac{s_{22}}{(s_{22}s_{55} - s_{25}^2)};\ e_{66}(t_f) = \frac{s_{33}}{(s_{33}s_{66} - s_{36}^2)}.$$

Integrating the above differential equations with terminal conditions and writing $\mathbf{T} = (t_f - t)$, time-to-go, we get expressions for $\mathbf{e}_{ij}$; this are given in Table A-I below.

Table A-1

$e_{66}(t_f) = e_{66}(t) - 1/r_{33}\int_t^{t_f} d\sigma = e_{66}(t) - \frac{(t_f - t)}{r_{33}};\ \Rightarrow e_{66}(t) = e_{66}(t_f) + \frac{(t_f - t)}{r_{33}};$ $\Rightarrow e_{66}(t) = \frac{s_{33}}{(s_{33}s_{66} - s_{36}^2)} + \frac{(t_f - t)}{r_{33}};\ \Rightarrow e_{66}(t) = \left[\frac{r_{33}s_{33} + (s_{33}s_{66} - s_{36}^2)\,T}{r_{33}(s_{33}s_{66} - s_{36}^2)}\right]$
$e_{56}(t_f) = e_{56}(t);\ \Rightarrow e_{56}(t) = e_{56}(t_f) = 0$
$e_{55}(t_f) = e_{55}(t) - 1/r_{22}\int_t^{t_f} d\sigma = e_{55}(t) - \frac{(t_f - t)}{r_{22}};\ \Rightarrow e_{55}(t) = e_{55}(t_f) + \frac{(t_f - t)}{r_{22}};$ $\Rightarrow e_{55}(t) = \frac{s_{22}}{(s_{22}s_{55} - s_{25}^2)} + \frac{(t_f - t)}{r_{22}};\ \Rightarrow e_{55}(t) = \left[\frac{r_{22}s_{22} + (s_{22}s_{55} - s_{25}^2)\,T}{r_{22}(s_{22}s_{55} - s_{25}^2)}\right]$
$e_{46}(t_f) = e_{46}(t);\ \Rightarrow e_{46}(t) = e_{46}(t_f) = 0$
$e_{45}(t_f) = e_{45}(t);\ \Rightarrow e_{45}(t) = e_{45}(t_f) = 0$

Table A-1 (contd)

$e_{44}(t_f)=e_{44}(t)-\frac{1}{r_{11}}\int_t^{t_f} d\sigma=e_{44}(t)-\frac{(t_f-t)}{r_{11}}$; ➔ $e_{44}(t)=e_{44}(t_f)+\frac{(t_f-t)}{r_{11}}$;   ➔ $e_{44}(t)=\frac{s_{11}}{(s_{11}s_{44}-s_{14}^2)}+\frac{(t_f-t)}{r_{11}}$; ➔ $e_{44}(t)=\left[\frac{r_{11}s_{11}+(s_{11}s_{44}-s_{14}^2)\,T}{r_{11}(s_{11}s_{44}-s_{14}^2)}\right]$
$e_{36}(t_f)=e_{36}(t)+\int_t^{t_f} e_{66}(\sigma)\,d\sigma=e_{36}(t)+\int_t^{t_f}\left[e_{66}(t_f)+\frac{1}{r_{33}}(t_f-\sigma)\right]d\sigma$ ➔ $e_{36}(t)=e_{36}(t_f)-e_{66}(t_f)(t_f-t)-\frac{(t_f-t)^2}{2r_{33}}$ ➔ $e_{36}(t)=-\frac{s_{36}}{(s_{33}s_{66}-s_{36}^2)}-\frac{s_{33}}{(s_{33}s_{66}-s_{36}^2)}(t_f-t)-\frac{(t_f-t)^2}{2r_{33}}$ ➔ $e_{36}(t)=-\left[\frac{2r_{33}s_{36}+2r_{33}s_{33}T+(s_{33}s_{66}-s_{36}^2)\,T^2}{2r_{33}(s_{33}s_{66}-s_{36}^2)}\right]$
$e_{35}(t_f)=e_{35}(t)+\int_t^{t_f} e_{56}(\sigma)\,d\sigma=e_{35}(t)$; ➔ $e_{35}(t)=e_{35}(t_f)=0$
$e_{34}(t_f)=e_{34}(t)+\int_t^{t_f} e_{46}(\sigma)\,d\sigma=e_{34}(t)$; ➔ $e_{34}(t)=e_{34}(t_f)=0$
$e_{33}(t_f)=e_{33}(t)+2\int_t^{t_f} e_{36}(\sigma)\,d\sigma$ ➔ $e_{33}(t_f)=e_{33}(t)+2\int_t^{t_f}\left[e_{36}(t_f)-e_{66}(t_f)(t_f-t)-\frac{1}{2r_{33}}(t_f-t)^2\right]d\sigma$;   ➔ $e_{33}(t)=e_{33}(t_f)-2e_{36}(t_f)(t_f-t)+e_{66}(t_f)(t_f-t)^2+\frac{1}{3r_{33}}(t_f-t)^3$;   ➔ $e_{33}(t)=\frac{s_{66}}{(s_{33}s_{66}-s_{36}^2)}+\frac{2s_{36}(t_f-t)}{(s_{33}s_{66}-s_{36}^2)}+\frac{s_{33}(t_f-t)^2}{(s_{33}s_{66}-s_{36}^2)}+\frac{(t_f-t)^3}{3r_{33}}$ ➔ $e_{33}(t)=\left[\frac{3r_{33}s_{66}+6r_{33}s_{36}T+3r_{33}s_{33}T^2+(s_{33}s_{66}-s_{36}^2)\,T^3}{3r_{33}(s_{33}s_{66}-s_{36}^2)}\right]$

$e_{26}(t_f)=e_{26}(t)+\int_t^{t_f} e_{56}(\sigma)d\sigma=e_{26}(t);\ \Rightarrow e_{26}(t)=e_{26}(t_f)=0$
$e_{25}(t_f)=e_{25}(t)+\int_t^{t_f} e_{55}(\sigma)d\sigma=e_{25}(t)+\int_t^{t_f}\left[e_{55}(t_f)+\frac{1}{r_{22}}(t_f-\sigma)\right]d\sigma;$ $\Rightarrow e_{25}(t)=e_{25}(t_f)-e_{55}(t_f)(t_f-t)-\frac{(t_f-t)^2}{2r_{22}}$ $\Rightarrow e_{25}(t)=-\frac{s_{25}}{\left(s_{22}s_{55}-s_{25}^2\right)}-\frac{s_{22}(t_f-t)}{\left(s_{22}s_{55}-s_{25}^2\right)}-\frac{(t_f-t)^2}{2r_{22}}$ $\Rightarrow e_{25}(t)=-\left[\frac{2r_{22}s_{25}+2r_{22}s_{22}T+\left(s_{22}s_{55}-s_{25}^2\right)T^2}{2r_{22}\left(s_{22}s_{55}-s_{25}^2\right)}\right]$
$e_{24}(t_f)=e_{24}(t)+\int_t^{t_f} e_{45}(\sigma)d\sigma=e_{24}(t);\ \Rightarrow e_{24}(t)=e_{24}(t_f)=0$
$e_{23}(t_f)=e_{23}(t)+\int_t^{t_f}\left[e_{26}(\sigma)+e_{35}(\sigma)\right]d\sigma=e_{23}(t);\ \Rightarrow\ x_{23}(t)=x_{23}(t_f)=0$
$e_{22}(t_f)=e_{22}(t)+\int_t^{t_f} 2e_{25}(\sigma)d\sigma$ $\Rightarrow e_{22}(t_f)=e_{22}(t)+\int_t^{t_f} 2\left[e_{25}(t_f)-e_{55}(t_f)(t_f-\sigma)-\frac{1}{2r_{22}}(t_f-\sigma)^2\right]d\sigma$ $\Rightarrow\ e_{22}(t)=e_{22}(t_f)-2e_{25}(t_f)(t_f-t)+e_{55}(t_f)(t_f-t)^2+\frac{(t_f-t)^3}{3r_{22}}$ $\Rightarrow\ e_{22}(t)=\frac{s_{55}}{\left(s_{22}s_{55}-s_{25}^2\right)}+\frac{2s_{25}(t_f-t)}{\left(s_{22}s_{55}-s_{25}^2\right)}+\frac{s_{22}(t_f-t)^2}{\left(s_{22}s_{55}-s_{25}^2\right)}+\frac{(t_f-t)^3}{3r_{22}}$ $\Rightarrow\ e_{22}(t)=\left[\frac{3r_{22}s_{55}+6r_{22}s_{25}T+3r_{22}s_{22}T^2+\left(s_{22}s_{55}-s_{25}^2\right)T^3}{3r_{22}\left(s_{22}s_{55}-s_{25}^2\right)}\right]$
$e_{16}(t_f)=e_{16}(t);\ \Rightarrow\ e_{16}(t)=e_{16}(t_f)=0$

Table A-1 (contd)

$$e_{15}(t_f) = e_{15}(t);\ \Rightarrow\ e_{15}(t) = e_{15}(t_f) = 0$$

$$e_{14}(t_f) = e_{14}(t) + \int_t^{t_f} e_{44}(\sigma)\,d\sigma = e_{14}(t) + \int_t^{t_f} \left[e_{44}(t_f) + \tfrac{1}{r}(t_f - \sigma) \right] d\sigma$$

$$\Rightarrow\ e_{14}(t_f) = e_{14}(t) + e_{44}(t_f)(t_f - t) + \frac{(t_f - t)^2}{2r_{11}}$$

$$\Rightarrow\ e_{14}(t) = -\frac{s_{14}}{\left(s_{11}s_{44} - s_{14}^2\right)} - \frac{s_{11}(t_f - t)}{\left(s_{11}s_{44} - s_{14}^2\right)} - \frac{(t_f - t)^2}{2r_{11}}$$

$$\Rightarrow\ e_{14}(t) = -\left[\frac{2r_{11}s_{14} + 2r_{11}s_{11}T + \left(s_{11}s_{44} - s_{14}^2\right) T^2}{2r_{11}\left(s_{11}s_{44} - s_{14}^2\right)} \right]$$

$$e_{13}(t_f) = e_{13}(t_f) + \int_t^{t_f} \left[e_{16}(\sigma) + e_{34}(\sigma) \right] d\sigma = e_{13}(t_f);\ \Rightarrow\ e_{13}(t) = e_{13}(t_f) = 0$$

$$e_{12}(t_f) = e_{12}(t_f) + \int_t^{t_f} \left[e_{15}(\sigma) + e_{24}(\sigma) \right] d\sigma = e_{12}(t_f);\ \Rightarrow\ e_{12}(t) = e_{12}(t_f) = 0$$

$$e_{11}(t_f) = e_{11}(t) + \int_t^{t_f} 2e_{14}(\sigma)\,d\sigma$$

$$\Rightarrow\ e_{11}(t_f) = e_{11}(t) + \int_t^{t_f} 2\left[e_{14}(t_f) - e_{44}(t_f)(t_f - \sigma) - \tfrac{1}{2r_{11}}(t_f - \sigma)^2 \right] d\sigma$$

$$\Rightarrow\ e_{11}(t) = e_{11}(t_f) - 2e_{14}(t_f)(t_f - t) + e_{44}(t_f)(t_f - t)^2 + \frac{(t_f - t)^3}{3r_{11}}$$

$$\Rightarrow\ e_{11}(t) = \frac{s_{44}}{\left(s_{11}s_{44} - s_{14}^2\right)} + \frac{2s_{14}(t_f - t)}{\left(s_{11}s_{44} - s_{14}^2\right)} + \frac{s_{11}(t_f - t)^2}{\left(s_{11}s_{44} - s_{14}^2\right)} + \frac{(t_f - t)^3}{3r_{11}}$$

$$\Rightarrow\ e_{11}(t) = \left[\frac{3r_{11}s_{44} + 6r_{11}s_{14}T + 3r_{11}s_{11}T^2 + \left(s_{11}s_{44} - s_{14}^2\right) T^3}{3r_{11}\left(s_{11}s_{44} - s_{14}^2\right)} \right]$$

In view of table A-1, we may write:

$$\mathbf{E}=\begin{bmatrix} e_{11} & 0 & 0 & e_{14} & 0 & 0 \\ 0 & e_{22} & 0 & 0 & e_{25} & 0 \\ 0 & 0 & e_{33} & 0 & 0 & e_{36} \\ e_{14} & 0 & 0 & e_{44} & 0 & 0 \\ 0 & e_{25} & 0 & 0 & e_{55} & 0 \\ 0 & 0 & e_{36} & 0 & 0 & e_{66} \end{bmatrix} \tag{A3-23}$$

Inversion of the $\mathbf{E}$ matrix to obtain $\mathbf{P}$ can proceed in the same way as, shown earlier for $\mathbf{S}$. Since $\mathbf{P}$ is a symmetric matrix, we write it as:

$$\mathbf{E}^{-1}=\mathbf{P}=\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12} & \mathbf{P}_{22} \end{bmatrix} \tag{A3-24}$$

Where:

$$\mathbf{E}_{11}=\begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix};\ \mathbf{E}_{12}=\begin{bmatrix} e_{14} & 0 & 0 \\ 0 & e_{25} & 0 \\ 0 & 0 & e_{36} \end{bmatrix};\ \mathbf{E}_{22}=\begin{bmatrix} e_{44} & 0 & 0 \\ 0 & e_{55} & 0 \\ 0 & 0 & e_{66} \end{bmatrix}$$

And:

$$\mathbf{E}_{11}^{-1}=\begin{bmatrix} \frac{1}{e_{11}} & 0 & 0 \\ 0 & \frac{1}{e_{22}} & 0 \\ 0 & 0 & \frac{1}{e_{33}} \end{bmatrix};\ \mathbf{E}_{12}^{-1}=\begin{bmatrix} \frac{1}{e_{14}} & 0 & 0 \\ 0 & \frac{1}{e_{25}} & 0 \\ 0 & 0 & \frac{1}{e_{36}} \end{bmatrix};\ \mathbf{S}_{22}^{-1}=\begin{bmatrix} \frac{1}{e_{44}} & 0 & 0 \\ 0 & \frac{1}{e_{55}} & 0 \\ 0 & 0 & \frac{1}{e_{66}} \end{bmatrix}$$

For $\mathbf{P}$ to be the inverse of $\mathbf{E}$, we must have:

$$\mathbf{E}\ \mathbf{P}=\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{12} & \mathbf{E}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12} & \mathbf{P}_{22} \end{bmatrix}=\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{A3-25}$$

Using expressions for $\mathbf{E}_{ij}$ and $\mathbf{E}_{ij}^{-1}$, it follows from (A3-25) that:

$$\mathbf{P}_{11}=\begin{bmatrix} \frac{e_{44}}{\left(e_{11}e_{44}-e_{14}^{2}\right)} & 0 & 0 \\ 0 & \frac{e_{55}}{\left(e_{22}e_{55}-e_{25}^{2}\right)} & 0 \\ 0 & 0 & \frac{e_{66}}{\left(e_{33}e_{66}-e_{36}^{2}\right)} \end{bmatrix} \tag{A3-26}$$

$$\mathbf{P_{12}} = \begin{bmatrix} \dfrac{-e_{14}}{\left(e_{11}e_{44} - e_{14}{}^{2}\right)} & 0 & 0 \\ 0 & \dfrac{-e_{25}}{\left(e_{22}e_{55} - e_{25}{}^{2}\right)} & 0 \\ 0 & 0 & \dfrac{-e_{36}}{\left(e_{33}e_{66} - e_{36}{}^{2}\right)} \end{bmatrix} \qquad \text{(A3-27)}$$

$$\mathbf{P_{22}} = \begin{bmatrix} \dfrac{e_{11}}{\left(e_{11}e_{44} - e_{14}{}^{2}\right)} & 0 & 0 \\ 0 & \dfrac{e_{22}}{\left(e_{22}e_{55} - e_{25}{}^{2}\right)} & 0 \\ 0 & 0 & \dfrac{e_{33}}{\left(e_{33}e_{66} - e_{36}{}^{2}\right)} \end{bmatrix} \qquad \text{(A3-28)}$$

Where:

$$\mathbf{P_{11}} = \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix};\ \mathbf{P_{12}} = \begin{bmatrix} p_{14} & 0 & 0 \\ 0 & p_{25} & 0 \\ 0 & 0 & p_{36} \end{bmatrix};\ \mathbf{P_{22}} = \begin{bmatrix} p_{44} & 0 & 0 \\ 0 & p_{55} & 0 \\ 0 & 0 & p_{66} \end{bmatrix}$$

Substituting for $\mathbf{e_{ij}}$, from Table A-I we can now derive expressions for $\mathbf{p_{ij}}$, using equations (A3-26) – (A3-28). A detailed derivation is shown below:

Now, $p_{11} = \dfrac{n_{11}}{d_{11}} = \dfrac{e_{44}}{\left(e_{11}e_{44} - e_{14}{}^{2}\right)}$

Where:

$$n_{11} = e_{44}(t) = \frac{\left\lfloor r_{11}s_{11} + \left(s_{11}s_{44} - s_{14}{}^{2}\right)T \right\rfloor}{r_{11}\left(s_{11}s_{44} - s_{14}{}^{2}\right)}$$

$$d_{11} = \left(e_{11}e_{44} - e_{14}{}^{2}\right)$$

$$= \left[\frac{3r_{11}s_{44} + 6r_{11}s_{14}T + 3r_{11}s_{11}T^{2} + \left(s_{11}s_{44} - s_{14}{}^{2}\right)T^{3}}{3r_{11}\left(s_{11}s_{44} - s_{14}{}^{2}\right)}\right]\left[\frac{r_{11}s_{11} + \left(s_{11}s_{44} - s_{14}{}^{2}\right)T}{r_{11}\left(s_{11}s_{44} - s_{14}{}^{2}\right)}\right]$$

$$- \left[\frac{2r_{11}s_{14} + 2r_{11}s_{11}T + \left(s_{11}s_{44} - s_{14}{}^{2}\right)T^{2}}{2r_{11}\left(s_{11}s_{44} - s_{14}{}^{2}\right)}\right]^{2}$$

➔

$$d_{11} = \frac{\left\{\begin{array}{l} 3r_{11}^2 s_{11}s_{44} + 3r_{11}s_{44}\left(r_{11}s_{44} - s_{14}^2\right)T + 6r_{11}^2 s_{11}s_{14}T + 6r_{11}s_{14}\left(r_{11}s_{44} - s_{14}^2\right)T^2 \\ + 3r_{11}^2 s_{11}^2 T^2 + 4r_{11}s_{11}\left(r_{11}s_{44} - s_{14}^2\right)T^3 + \left(r_{11}s_{44} - s_{14}^2\right)^2 T^4 \end{array}\right\}}{3r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right)^2} \cdots$$

$$- \frac{\left\{\begin{array}{l} 4r_{11}^2 s_{14}^2 + 4r_{11}^2 s_{11}^2 T^2 + \left(s_{11}s_{44} - s_{14}^2\right)^2 T^4 + 8r_{11}^2 s_{11}s_{14}T \\ + 4r_{11}s_{14}\left(s_{11}s_{44} - s_{14}^2\right)T^2 + 4r_{11}s_{11}\left(s_{11}s_{44} - s_{14}^2\right)T^3 \end{array}\right\}}{4r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right)^2}$$

➔

$$d_{11} = \frac{\left\{\begin{array}{l} 12r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right) + 12r_{11}s_{44}\left(s_{11}s_{44} - s_{14}^2\right)T + 12r_{11}s_{14}\left(r_{11}s_{44} - s_{14}^2\right)T^2 \\ + 4r_{11}s_{11}\left(r_{11}s_{44} - s_{14}^2\right)T^3 + \left(s_{11}s_{44} - s_{14}^2\right)^2 T^4 \end{array}\right\}}{12r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right)^2}$$

➔

$$p_{11} = \frac{n_{11}}{d_{11}} = \frac{12r_{11}\left[r_{11}s_{11} + \left(s_{11}s_{44} - s_{14}^2\right)T\right]}{\left[12r_{11}^2 + 12r_{11}s_{44}\,T + 12r_{11}s_{14}\,T^2 + 4r_{11}s_{11}\,T^3 + \left(s_{11}s_{44} - s_{14}^2\right)T^4\right]} \qquad \text{(A3-29)}$$

By proceeding in the same manner as above it can be shown that:

$$p_{22} = \frac{12r_{22}\left[r_{22}s_{22} + \left(s_{22}s_{55} - s_{25}^2\right)T\right]}{\left[12r_{22}^2 + 12r_{22}s_{55}\,T + 12r_{22}s_{25}\,T^2 + 4r_{22}s_{22}\,T^3 + \left(s_{22}s_{55} - s_{25}^2\right)T^4\right]} \qquad \text{(A3-30)}$$

$$p_{33} = \frac{12r_{33}\left[r_{33}s_{33} + \left(s_{33}s_{66} - s_{36}^2\right)T\right]}{\left[12r_{33}^2 + 12r_{33}s_{66}\,T + 12r_{33}s_{36}\,T^2 + 4r_{33}s_{33}\,T^3 + \left(s_{33}s_{66} - s_{36}^2\right)T^4\right]} \qquad \text{(A3-31)}$$

Now:

$$p_{44} = \frac{n_{44}}{d_{44}} = \frac{e_{11}}{\left(e_{11}e_{44} - e_{14}^2\right)}$$

Where:

$$n_{44} = \left[\frac{3r_{11}s_{44} + 6r_{11}s_{14}T + 3r_{11}s_{11}T^2 + \left(s_{11}s_{44} - s_{14}^2\right)T^3}{3r_{11}\left(s_{11}s_{44} - s_{14}^2\right)}\right]$$

$$d_{44} = \frac{\left\{\begin{array}{l} 12r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right) + 12r_{11}s_{44}\left(s_{11}s_{44} - s_{14}^2\right)T + 12r_{11}s_{14}\left(r_{11}s_{44} - s_{14}^2\right)T^2 \\ + 4r_{11}s_{11}\left(r_{11}s_{44} - s_{14}^2\right)T^3 + \left(s_{11}s_{44} - s_{14}^2\right)^2 T^4 \end{array}\right\}}{12r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right)^2}$$

➔

$$p_{44} = \frac{4r_{11}\left[3r_{11}s_{44} + 6r_{11}s_{14}T + 3r_{11}s_{11}T^2 + \left(s_{11}s_{44} - s_{14}^2\right) T^3\right]}{\left[12r_{11}^2 + 12r_{11}s_{44}\ T + 12r_{11}s_{14}\ T^2 + 4r_{11}s_{11}\ T^3 + \left(s_{11}s_{44} - s_{14}^2\right)\ T^4\right]} \quad \text{(A3-32)}$$

Similarly, it can be shown that:

$$p_{55} = \frac{4r_{22}\left[3r_{22}s_{55} + 6r_{22}s_{25}T + 3r_{22}s_{22}T^2 + \left(s_{22}s_{55} - s_{25}^2\right) T^3\right]}{\left[12r_{22}^2 + 12r_{22}s_{55}\ T + 12r_{22}s_{25}\ T^2 + 4r_{22}s_{22}\ T^3 + \left(s_{22}s_{55} - s_{25}^2\right)\ T^4\right]} \quad \text{(A3-33)}$$

$$p_{66} = \frac{4r_{33}\left[3r_{33}s_{66} + 6r_{33}s_{36}T + 3r_{33}s_{33}T^2 + \left(s_{33}s_{66} - s_{36}^2\right) T^3\right]}{\left[12r_{33}^2 + 12r_{33}s_{66}\ T + 12r_{33}s_{36}\ T^2 + 4r_{33}s_{55}\ T^3 + \left(s_{33}s_{66} - s_{36}^2\right)\ T^4\right]} \quad \text{(A3-34)}$$

Further:

$$p_{14} = \frac{n_{14}}{d_{14}} = \frac{-e_{14}}{\left(e_{11}e_{44} - e_{14}^2\right)}$$

Where:

$$n_{14} = \left[\frac{2r_{11}s_{14} + 2r_{11}s_{11}T + \left(s_{11}s_{44} - s_{14}^2\right) T^2}{2r_{11}\left(s_{11}s_{44} - s_{14}^2\right)}\right]$$

$$d_{14} = \frac{\left\{\begin{array}{l} 12r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right) + 12r_{11}s_{44}\left(s_{11}s_{44} - s_{14}^2\right) T + 12r_{11}s_{14}\left(r_{11}s_{44} - s_{14}^2\right) T^2 \cdots \\ + 4r_{11}s_{11}\left(r_{11}s_{44} - s_{14}^2\right) T^3 + \left(s_{11}s_{44} - s_{14}^2\right)^2 T^4 \end{array}\right\}}{12r_{11}^2\left(s_{11}s_{44} - s_{14}^2\right)^2}$$

➔

$$p_{14} = \frac{6r_{11}\left[2r_{11}s_{14} + 2r_{11}s_{11}T + \left(s_{11}s_{44} - s_{14}^2\right) T^2\right]}{\left[12r_{11}^2 + 12r_{11}s_{44}\ T + 12r_{11}s_{14}\ T^2 + 4r_{11}s_{11}\ T^3 + \left(s_{11}s_{44} - s_{14}^2\right)\ T^4\right]} \quad \text{(A3-35)}$$

Similarly, it can be shown that:

$$p_{25} = \frac{6r_{22}\left[2r_{22}s_{25} + 2r_{22}s_{22}T + \left(s_{22}s_{55} - s_{25}^2\right) T^2\right]}{\left[12r_{22}^2 + 12r_{22}s_{55}\ T + 12r_{22}s_{25}\ T^2 + 4r_{22}s_{22}\ T^3 + \left(s_{22}s_{55} - s_{25}^2\right)\ T^4\right]} \quad \text{(A3-36)}$$

$$p_{36} = \frac{6r_{33}\left[2r_{33}s_{36} + 2r_{33}s_{33}T + \left(s_{33}s_{66} - s_{36}^2\right) T^2\right]}{\left[12r_{33}^2 + 12r_{33}s_{66}\ T + 12r_{33}s_{36}\ T^2 + 4r_{33}s_{33}\ T^3 + \left(s_{33}s_{66} - s_{36}^2\right)\ T^4\right]} \quad \text{(A3-37)}$$

A.4 Solution of the Vector Riccati Deferential Equation

Let us now consider equation (A2.5), that is:

$$\dot{\underline{\xi}} = -\left[A - BR^{-1}B^T P\right]^T \underline{\xi} - PB\left(\underline{a}_1^d - \underline{a}_2^d\right) \quad \text{(A4-1)}$$

Now:

$$\left[\mathbf{A}-\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{T}\mathbf{P}\right]^{T}=\begin{bmatrix} 0 & 0 & 0 & \frac{-p_{14}}{r_{11}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-p_{25}}{r_{22}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-p_{36}}{r_{33}} \\ 1 & 0 & 0 & \frac{-p_{44}}{r_{11}} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-p_{55}}{r_{22}} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{-p_{66}}{r_{33}} \end{bmatrix};\ \mathbf{PB}=\begin{bmatrix} p_{14} & 0 & 0 \\ 0 & p_{25} & 0 \\ 0 & 0 & p_{36} \\ p_{44} & 0 & 0 \\ 0 & p_{55} & 0 \\ 0 & 0 & p_{66} \end{bmatrix} \tag{A4-2}$$

Writing: $\underline{\xi}=[\xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4 \quad \xi_5 \quad \xi_6]^T$; equation (A4-1) (in its decomposed form) may be written as:

$$\dot{\xi}_1=\frac{p_{14}}{r_{11}}\xi_4-p_{14}\left(a^d_{x_1}-a^d_{x_2}\right) \tag{A4-3}$$

$$\dot{\xi}_2=\frac{p_{25}}{r_{22}}\xi_5-p_{25}\left(a^d_{y_1}-a^d_{y_2}\right) \tag{A4-4}$$

$$\dot{\xi}_3=\frac{p_{36}}{r_{33}}\xi_6-p_{36}\left(a^d_{z_1}-a^d_{z_2}\right) \tag{A4-5}$$

$$\dot{\xi}_4=-\xi_1+\frac{p_{44}}{r_{11}}\xi_4-p_{44}\left(a^d_{x_1}-a^d_{x_2}\right) \tag{A4-6}$$

$$\dot{\xi}_5=-\xi_2+\frac{p_{55}}{r_{22}}\xi_5-p_{55}\left(a^d_{y_1}-a^d_{y_2}\right) \tag{A4-7}$$

$$\dot{\xi}_6=-\xi_3+\frac{p_{66}}{r_{33}}\xi_6-p_{66}\left(a^d_{z_1}-a^d_{z_2}\right) \tag{A4-8}$$

Unfortunately it is not easily possible to obtain analytical solutions to equations (A4-4) -(A4-9), except for special cases where $\left(a^d_{x_i}, a^d_{x_i}, a^d_{x_i}\right)$, $\left(a^d_{y_i}, a^d_{y_i}, a^d_{z_i}\right)$ and $\left(a^d_{z_i}, a^d_{z_i}, a^d_{z_i}\right)$, $i=1,2$ are constants. This case will be considered later on in this appendix. In general, equations (A4-4) – (A4-9) have to be solved backwards in time. For this purpose we make the substitutions:

Let: $T=t_f-t$, $\Rightarrow dT=-dt$; $\underline{\xi}(t)=\underline{\xi}(t_f-T)=\underline{\eta}(T)$; $a^d_{\gamma_i}(t)=a^d_{\gamma_i}(t_f-T)=\alpha^d_{\gamma_i}(T)$; $i=1,2$; $\gamma=x,y,z$. Hence the above equations (A4-3) – (A4-8) may be written as:

$$-\frac{d\eta_1}{dT}=\frac{p_{14}}{r_{11}}\eta_4-p_{14}\left(\alpha^d_{x_1}-\alpha^d_{x_2}\right) \tag{A4-9}$$

$$-\frac{d\eta_2}{dT}=\frac{p_{25}}{r_{22}}\eta_5-p_{25}\left(\alpha^d_{y_1}-\alpha^d_{y_2}\right) \tag{A4-10}$$

$$-\frac{d\eta_3}{dT}=\frac{p_{36}}{r_{33}}\eta_6-p_{36}\left(\alpha^d_{z_1}-\alpha^d_{z_2}\right) \tag{A4-11}$$

$$-\frac{d\eta_4}{dT} = -\eta_1 + \frac{p_{44}}{r_{11}}\eta_4 - p_{44}\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) \tag{A4-12}$$

$$-\frac{d\eta_5}{dT} = -\eta_2 + \frac{p_{55}}{r_{22}}\eta_5 - p_{55}\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) \tag{A4-13}$$

$$-\frac{d\eta_6}{dT} = -\eta_3 + \frac{p_{66}}{r_{33}}\eta_6 - p_{66}\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \tag{A4-14}$$

These equations satisfy the boundary condition that $\underline{\eta}(0) = \underline{\xi}(t_f) = \underline{0}$, and must be solved backwards in time, that is, $T \to 0$. We shall regard $\underline{\eta}$ as time-to-go equivalent of $\underline{\xi}$.

A.4.1. Analytic Solution of the VRDE – Special Case

Analytical solution of the VRDE is possible for the case when: $s_{11} = s_{22} = s_{33} = 2s_1$; $s_{14} = s_{25} = s_{36} = s_2 = 0$; $s_{44} = s_{55} = s_{66} = 2s_3 = 0$ and $r_{11} = r_{22} = r_{33} = 2r$. For this case (see section4.1):

$$p_{11} = p_{22} = p_{33} = \frac{6rs_1}{\left[3r + s_1 T^3\right]} \tag{A4-15}$$

$$p_{14} = p_{25} = p_{26} = \frac{6rs_1 T}{\left[3r + s_1 T^3\right]} \tag{A4-16}$$

$$p_{44} = p_{55} = p_{66} = \frac{6rs_1 T^2}{\left[3r + s_1 T^3\right]} \tag{A4-17}$$

Multiplying both sides of equations (A4-9)-(A4-11) respectively by $\left(\frac{p_{44}}{p_{14}}\right)$, $\left(\frac{p_{55}}{p_{25}}\right)$, $\left(\frac{p_{66}}{p_{36}}\right)$; we get:

$$-\left(\frac{p_{44}}{p_{14}}\right)\frac{d\eta_1}{dT} = \frac{p_{44}}{2r}\eta_4 - p_{44}\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) \tag{A4-18}$$

$$-\left(\frac{p_{55}}{p_{25}}\right)\frac{d\eta_2}{dT} = \frac{p_{55}}{2r}\eta_5 - p_{55}\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) \tag{A4-19}$$

$$-\left(\frac{p_{44}}{p_{14}}\right)\frac{d\eta_3}{dT} = \frac{p_{66}}{2r}\eta_6 - p_{66}\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \tag{A4-20}$$

$$-\frac{d\eta_4}{dT} = -\eta_1 + \frac{p_{44}}{2r}\eta_4 - p_{44}\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) \tag{A4-21}$$

$$-\frac{d\eta_5}{dT} = -\eta_2 + \frac{p_{55}}{2r}\eta_5 - p_{55}\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) \tag{A4-22}$$

$$-\frac{d\eta_6}{dT} = -\eta_3 + \frac{p_{66}}{2r}\eta_6 - p_{66}\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \tag{A4-23}$$

Subtracting equations (A4-18) – (A4-20) respectively from equations (A4-21) – (A4-23) and rearranging the terms, we get:

$$\frac{d\eta_4}{dT} = \eta_1 + \left(\frac{p_{44}}{p_{14}}\right)\frac{d\eta_1}{dT} = \eta_1 + T\frac{d\eta_1}{dT} = \frac{d}{dT}\left(T\eta_1\right)$$

$$\frac{d\eta_5}{dT} = \eta_2 + \left(\frac{p_{55}}{p_{25}}\right)\frac{d\eta_2}{dT} = \eta_2 + T\frac{d\eta_2}{dT} = \frac{d}{dT}(T\eta_2)$$
$$\frac{d\eta_5}{dT} = \eta_3 + \left(\frac{p_{66}}{p_{36}}\right)\frac{d\eta_3}{dT} = \eta_3 + T\frac{d\eta_3}{dT} = \frac{d}{dT}(T\eta_3)$$

which gives us:

$$\eta_4 = T\eta_1;\ \eta_5 = T\eta_2;\ \eta_6 = T\eta_3 \qquad \text{(A4-24)}$$

Substituting from equation (A4-24) back into equations (A4-21) - (A4-23), substituting for p_{ij} from (A4-16) - (A4-18) gives us:

$$\frac{d\eta_1}{dT} = -\frac{3s_1T^2}{\left[3r + s_1T^3\right]}\eta_1 + \frac{6rs_1T}{\left[3r + s_1T^3\right]}\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) \qquad \text{(A4-25)}$$
$$\frac{d\eta_2}{dT} = -\frac{3s_1T^2}{\left[3r + s_1T^3\right]}\eta_2 + \frac{6rs_1T}{\left[3r + s_1T^3\right]}\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) \qquad \text{(A4-26)}$$
$$\frac{d\eta_3}{dT} = -\frac{3s_1T^2}{\left[3r + s_1T^3\right]}\eta_3 + \frac{6rs_1T}{\left[3r + s_1T^3\right]}\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \qquad \text{(A4-27)}$$

➔

$$\frac{d}{dT}\left[\left(3r + s_1T^3\right)\eta_1\right] = 6rs_1T\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) = \left(\alpha^d_{x_1} - \alpha^d_{x_2}\right)\frac{d}{dT}\left(3rs_1T^2\right) \qquad \text{(A4-28)}$$
$$\frac{d}{dT}\left[\left(3r + s_1T^3\right)\eta_2\right] = 6rs_1T\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) = \left(\alpha^d_{y_1} - \alpha^d_{y_2}\right)\frac{d}{dT}\left(3rs_1T^2\right) \qquad \text{(A4-29)}$$
$$\frac{d}{dT}\left[\left(3r + s_1T^3\right)\eta_3\right] = 6rs_1T\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) = \left(\alpha^d_{z_1} - \alpha^d_{z_2}\right)\frac{d}{dT}\left(3rs_1T^2\right) \qquad \text{(A4-30)}$$

Assuming $\alpha_{x_i}, \alpha_{y_i}, \alpha_{z_i}$ are constants; equations (A4-28)-(A4-30) give us:

$$\eta_1 = \left[\frac{3rs_1T^2}{3r + s_1T^3}\right]\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) \qquad \text{(A4-31)}$$
$$\eta_2 = \left[\frac{3rs_1T^2}{3r + s_1T^3}\right]\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) \qquad \text{(A4-32)}$$
$$\eta_3 = \left[\frac{3rs_1T^2}{3r + s_1T^3}\right]\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \qquad \text{(A4-33)}$$

And it follows from (A4-24) that:

$$\eta_4 = \left[\frac{3rs_1T^3}{3r + s_1T^3}\right]\left(\alpha^d_{x_1} - \alpha^d_{x_2}\right) \qquad \text{(A4-34)}$$
$$\eta_5 = \left[\frac{3rs_1T^3}{3r + s_1T^3}\right]\left(\alpha^d_{y_1} - \alpha^d_{y_2}\right) \qquad \text{(A4-35)}$$
$$\eta_6 = \left[\frac{3rs_1T^3}{3r + s_1T^3}\right]\left(\alpha^d_{z_1} - \alpha^d_{z_2}\right) \qquad \text{(A4-36)}$$

Finally, the disturbance term in the feedback guidance may be written as:

$$\mathbf{K}_1^d \underline{\eta} = \mathbf{R}_1^{-1}\mathbf{B}^T \underline{\eta} = \frac{1}{2r_1}\begin{bmatrix}\eta_4\\ \eta_5\\ \eta_6\end{bmatrix} = \frac{1}{2r_1}\left[\frac{3rs_1T^3}{3r+2s_1T^3}\right]\begin{bmatrix}\left(\alpha_{x_1}^d - \alpha_{x_2}^d\right)\\ \left(\alpha_{y_1}^d - \alpha_{y_2}^d\right)\\ \left(\alpha_{z_1}^d - \alpha_{z_2}^d\right)\end{bmatrix} \quad \text{(A4-37)}$$

$$\mathbf{K}_2^d \underline{\eta} = \mathbf{R}_2^{-1}\mathbf{B}^T \underline{\eta} = \frac{1}{2r_2}\begin{bmatrix}\eta_4\\ \eta_5\\ \eta_6\end{bmatrix} = \frac{1}{2r_2}\left[\frac{3rs_1T^3}{3r+2s_1T^3}\right]\begin{bmatrix}\left(\alpha_{x_1}^d - \alpha_{x_2}^d\right)\\ \left(\alpha_{y_1}^d - \alpha_{y_2}^d\right)\\ \left(\alpha_{z_1}^d - \alpha_{z_2}^d\right)\end{bmatrix} \quad \text{(A4-38)}$$

A.5 Sight Line Rates for Small Angles and Rates

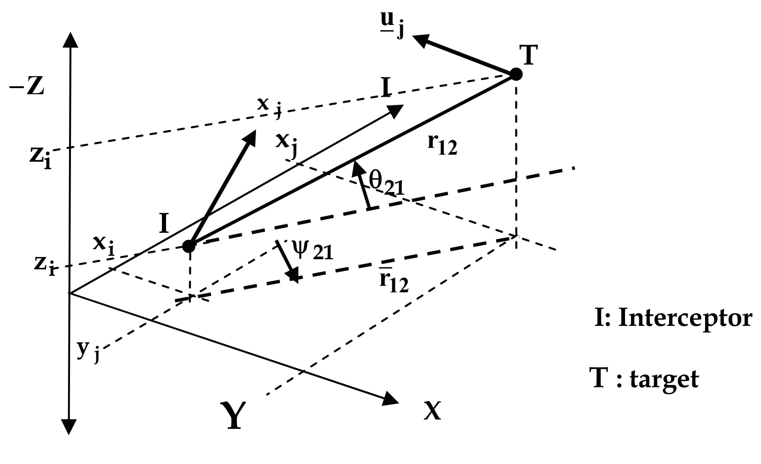

Figure A-1- Interceptor/Target Engagement Geometry

In order to establish connection with between the optimal guidance and the PN and APN we shall assume that the engagement trajectory is such that the azimuth and elevation sightline angles (ψ_{21}, θ_{21}) (Figure A-1) remain small during engagement, that is, the trajectory remains close to collision close geometry. For this condition it follows that the interceptor/target relative velocity is pointed approximately along the sight line and is approximately equal to the closing velocity $\mathbf{V_c}$. Under these conditions:

$$\mathbf{x}_{12} \approx \mathbf{r}_{12} \approx \mathbf{V_c T}\ ;\ \frac{\mathbf{d}}{\mathbf{dt}}\mathbf{x}_{12} = \mathbf{u}_{12} \approx -\mathbf{V_c} \quad \text{(A5-1)}$$

Where:

$\mathbf{r_{12} = \left(x_{12}{}^2 + y_{12}{}^2 + z_{12}{}^2\right)^{\frac{1}{2}}}$ **:** is the separation range between the interceptor and the target. Interceptor is designated the subscript (1) and the target is designated the subscript (2).

Also:

$$\mathbf{\bar{r}_{12} \approx r_{12} \approx V_c T\ ;\ \frac{d}{dt}\bar{r}_{12} \approx -V_c} \qquad \text{(A5-2)}$$

Where:

$\mathbf{\bar{r}_{12} = \left(x_{12}{}^2 + y_{12}{}^2\right)^{\frac{1}{2}}}$ **:** is the projection of separation range on to the x-y plane.

We shall first derive expressions for sightline rates of the interceptor (1) w.r.t the target (2) and then proceed to convert these to sightline rates of the target w.r.t the interceptor; where;

$\left(\psi_{12}, \theta_{12}\right)$ are respectively the azimuth and elevation sightline angles of the interceptor (1) w.r.t the target (2).

$\left(\psi_{21} = -\psi_{21},\ \ \theta_{12} = -\theta_{21}\right)$ are respectively the azimuth and elevation sightline angles of the target (2) w.r.t interceptor (1).

$$\mathbf{\tan\psi_{12} = \frac{y_{12}}{x_{12}}\ ;\ \Rightarrow\ \dot{\psi}_{12}\sec^2\psi_{12} = \left(\frac{\dot{y}_{12}}{x_{12}} - \frac{y_{12}\,\dot{x}_{12}}{x_{12}{}^2}\right);\Rightarrow \dot{\psi}_{12} = \left(\frac{\dot{y}_{12}\,x_{12}}{r_{12}{}^2} - \frac{y_{12}\,\dot{x}_{12}}{r_{12}{}^2}\right);}$$

$\Rightarrow$

$$\mathbf{\dot{\psi}_{12} = \left(\frac{v_{12}}{V_c T} + \frac{y_{12}}{V_c T^2}\right);\ \Rightarrow\quad \dot{\psi}_{21} = -\dot{\psi}_{12} = -\left(\frac{v_{12}}{V_c T} + \frac{y_{12}}{V_c T^2}\right)} \qquad \text{(A5-3)}$$

$$\mathbf{\tan\theta_{12} = \frac{z_{12}}{\bar{r}_{12}}\ ;\ \Rightarrow \dot{\theta}_{12}\sec^2\theta_{12} = \left(\frac{\dot{z}_{12}}{\bar{r}_{12}} - \frac{z_{12}\,\dot{\bar{r}}_{12}}{\bar{r}_{12}{}^2}\right);\ \Rightarrow\ \dot{\theta}_{12} = \left(\frac{\dot{z}_{12}\,\bar{r}_{12}}{r_{12}{}^2} - \frac{z_{12}\,\dot{\bar{r}}_{12}}{r_{12}{}^2}\right);}$$

$\Rightarrow$

$$\mathbf{\dot{\theta}_{12} = \left(\frac{w_{12}}{V_c T} + \frac{z_{12}}{V_c T^2}\right);\ \Rightarrow\quad \dot{\theta}_{21} = -\dot{\theta}_{12} = -\left(\frac{w_{12}}{V_c T} + \frac{z_{12}}{V_c T^2}\right)} \qquad \text{(A5-4)}$$

It also follows from equations (A5-1) that the expression:

$$\mathbf{\frac{1}{V_c T^2}x_{12} + \frac{1}{V_c T}u_{12} = \frac{1}{T} - \frac{1}{T} = 0} \qquad \text{(A5-5)}$$

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19. ABSTRACT

In this report the application of the differential game theory to missile guidance problem is considered. Interceptor/target relative kinematics equations for a 3-D engagement are derived in state space form suitable for implementing the feedback guidance through minimisation-maximisation of the performance index. This performance index is a generalisation of that utilised by previous researchers in this field and includes, in addition to the miss-distance term, other terms involving interceptor/target relative velocity terms. This latter inclusion allows the designer to influence the engagement trajectories so as to aid both the intercept and evasion strategies. Closed form expressions are derived for the Riccati differential equations and the feedback guidance gains. A link between the differential game theory based guidance and the optimal guidance, the proportional navigation (PN) and the augmented PN (APN) guidance is established.